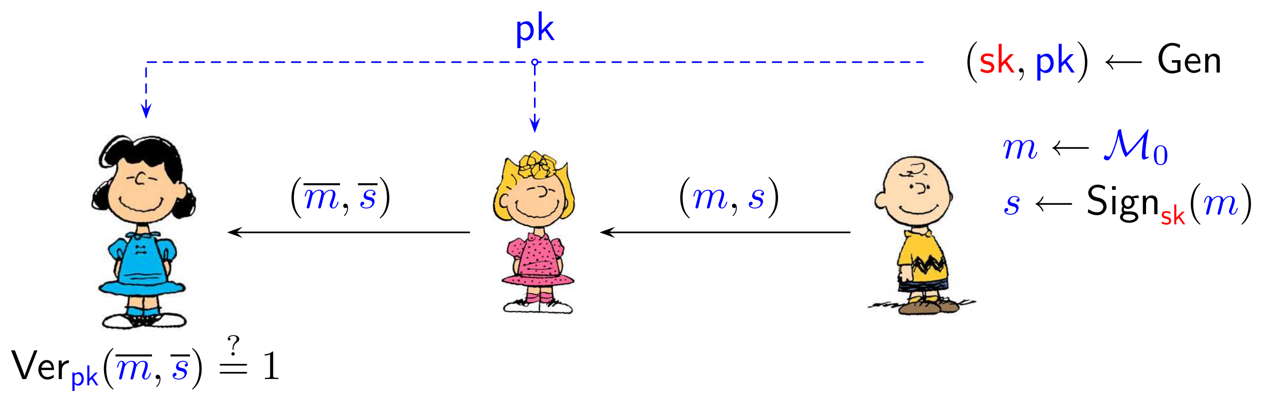
MTAT.07.003 Cryptology II

Digital Signatures

Sven Laur
University of Tartu

Formal Syntax

Digital signature scheme


pk
$(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$
$m \leftarrow \mathcal{M}_0$
$s \leftarrow \mathsf{Sign}_{\mathsf{sk}}(m)$
(m, s)
$(\overline{m}, \overline{s})$
$\mathsf{Ver}_{\mathsf{pk}}(\overline{m}, \overline{s}) \overset{?}{=} 1$

- ▷ To establish electronic identity, Charlie must generate $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ and convinces others that the public information pk represents him.
- ▷ A *keyed hash function* $\mathsf{Sign}_{\mathsf{sk}} : \mathcal{M} \rightarrow \mathcal{S}$ takes in a *plaintext* and outputs a corresponding *digital signature*.
- ▷ A *public verification algorithm* $\mathsf{Ver}_{\mathsf{pk}} : \mathcal{M} \times \mathcal{S} \rightarrow \{0, 1\}$ tries to distinguish between altered and original message pairs.
- ▷ The signature scheme is *functional* if for all $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ and $m \in \mathcal{M}$:

$$\mathsf{Ver}_{\mathsf{pk}}(m, \mathsf{Sign}_{\mathsf{sk}}(m)) = 1 \ .$$

Example. RSA-1024 signature scheme

Key generation Gen:

1. Choose uniformly 512-bit prime numbers p and q.
2. Compute $N = p \cdot q$ and $\phi(N) = (p-1)(q-1)$.
3. Choose uniformly $e \leftarrow \mathbb{Z}^*_{\phi(N)}$ and set $d = e^{-1} \mod \phi(N)$.
4. Output sk $= (p, q, e, d)$ and pk $= (N, e)$.

Signing and verification:

$$\mathcal{M} = \mathbb{Z}_N, \quad \mathcal{S} = \mathbb{Z}_N, \quad \mathcal{R} = \emptyset$$

$$\mathsf{Sign}_{\mathsf{sk}}(m) = m^d \mod N$$

$$\mathsf{Ver}_{\mathsf{pk}}(m, s) = 1 \quad \Leftrightarrow \quad m = s^e \mod N \ .$$

Security definitions

Attack scenarios

▷ *Key only attack*. The adversary has access only to the public key pk.

▷ *Chosen message attack.* Besides the public key pk, the adversary can adaptively query a list of valid signatures $(m_1, s_1), \ldots, (m_n, s_n)$.

Attack types

▷ *Universal forgery.* The adversary must create a valid signature for a prescribed message m that is chosen from a distribution $\mathcal{M}_0$.

▷ *Existential forgery.* The adversary must create a valid signature for some message m, i.e., there are no limitations on the choice of message.

$$\boxed{\text{One more signature attack} = \text{Chosen message attack} + \text{Existential forgery}}$$

Security against one more signature attack

A signature scheme is (t, q, ε)-*secure against one more signature* attack if

$$\mathsf{Adv}^{\mathsf{forge}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon$$

for any t-time adversary $\mathcal{A}$ where

```
$\mathcal{G}^{\mathcal{A}}$
    $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$
    For $i \in \{1, \ldots, q\}$ do
        $m_i \leftarrow \mathcal{A}(s_{i-1})$
        $s_i \leftarrow \mathsf{Sign}_{\mathsf{sk}}(m_i)$
    $(m, s) \leftarrow \mathcal{A}$
    if $(m, s) \in \{(m_1, s_1), \ldots, (m_q, s_q)\}$ **return** $0$
    **return** $\mathsf{Ver}_{\mathsf{pk}}(m, s)$
```

A conceptual description of digital signatures

A digital signature is a *non-interactive* and *transferable* counterpart of the following extended identification protocol:

1. Verifier sends a message m to a prover.
2. The prover accepts the message m.
3. The prover authenticates him or herself by using sk.

Transferability means that signature must be verifiable by other parties that did not participate in the creation of the signature.

As a result, digital signature is either

▷ a non-interactive proof of possession that is linked to the message m.

▷ a non-interactive proof of knowledge that is linked to the message m.

Digital Signatures Based on Proofs of Possession

Lamport one-time signatures

Public parameters:

Let $\{0,1\}^n$ be the message space and $f : \mathcal{X} \to \mathcal{Y}$ a one-way function.

Key Generation:

Generate $2 \times n$ random elements $x_{ij} \xleftarrow[u]{} \mathcal{X}$ and compute $y_{ij} \leftarrow f(x_{ij})$.
The secret key is sk $= (x_{ij})$ and the public key is pk $= (y_{ij})$.

Signing:

To sign a message $m = m_n \ldots m_1$ release $\sigma = (x_{1m_1}, \ldots, x_{nm_n})$.

Verification:

A signature (m, σ) is valid if $f(\sigma_i) = y_{im_i}$ for $i = 1, \ldots, n$.

Security of one-time signatures

Theorem. If f is (t, ε)-one-way function, then Lamport signature scheme is $(t, 1, 2n\varepsilon)$-secure against one more signature attack.

Proof. A successful forgery must reveal an inverse of $y_{i\neg m_i}$ for some i.

```
$\mathcal{B}(y)$
    Choose $2n-1$ elements of $(x_{ij}) \xleftarrow{u} \mathcal{X}$.
    Compute $2n-1$ entries $y_{ij} \leftarrow f(x_{ij})$.
    Put $y$ to the missing place and send pk to $\mathcal{A}$.
    Reveal $n$ elements of $(x_{ij})$ to $\mathcal{A}$ if possible.
    Return $x$ if $\mathcal{A}$ reveals $x$ such that $f(x) = y$.
```

Merkle signature scheme

$$\begin{tikzcd}
 & & & c_* & & & \\
 & c_0 \arrow[urr] & & & & c_1 \arrow[ull] & \\
y_{00} \arrow[ur] & & y_{01} \arrow[ul] & & y_{10} \arrow[ur] & & y_{11} \arrow[ul] \\
x_{00} \arrow[u] & & \underline{x_{01}} \arrow[u] & & x_{10} \arrow[u] & & x_{11} \arrow[u]
\end{tikzcd}$$

General description

▷ Public key pk is a root hash c_*

▷ Secret key sk consists of all leafs x_{ij}.

▷ A signature σ is a minimal amount of information needed to recompute c_*.

▷ A secret key can be compressed by a pseudoramndom function family.

Detailed description

▷ All intermediate values c_u are computed by hashing $c_u \leftarrow h(c_{u0}, c_{u1})$.

▷ The second level values y_u are computed as before $y_u \leftarrow f(x_u)$.

▷ The secret key can be further compacted by computing $x_u \leftarrow g(u, k)$ where $g : \mathcal{I} \times \mathcal{K} \to \mathcal{X}$ induces a pseudorandom function family $\mathcal{G} = \{g_k\}$.

Trapdoor one-way permutations

A collection of trapdoor permutations $\mathcal{F}_{\mathrm{tp}}$ is determined by three algorithms $(\mathsf{Gen}, \mathrm{Map}, \mathrm{Inv})$ such that

$$\forall(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}, \quad \forall m \in \mathcal{M}_{\mathsf{pk}} : \qquad \mathrm{Inv}_{\mathsf{sk}}(\mathrm{Map}_{\mathsf{pk}}(m)) = m$$

and both algorithms $\mathrm{Map}_{\mathsf{pk}}(\cdot)$ and $\mathrm{Inv}_{\mathsf{sk}}(\cdot)$ are deterministic.

OW-CPA security

A collection of trapdoor permutations $\mathcal{F}_{\mathrm{tp}}$ is (t,ε)-secure if for any t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\text{inv-cpa}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon$$

where

```
$\mathcal{G}^{\mathcal{A}}$
  $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$
  $m \xleftarrow[u]{} \mathcal{M}_{\mathsf{pk}}$
  $y \leftarrow \mathrm{Map}_{\mathsf{pk}}(m)$
  return $[\mathcal{A}(\mathsf{pk}, y) \overset{?}{=} m]$
```

Full Domain Hash

Setup

Run $(\mathsf{pk}, \mathsf{sk}) \leftarrow$ Gen to create an instance of trapdoor permutation. Choose $h : \mathcal{M} \rightarrow \mathcal{M}_{\mathsf{pk}}$ from a collision resistant function family $\mathcal{H}$.

Signing

To sign $m \in \mathcal{M}$, compute $x \leftarrow h(m)$ and output $s \leftarrow \mathrm{Inv}_{\mathsf{sk}}(x)$.

Verification

A pair (m, s) is a valid signature if $h(m) = \mathrm{Map}_{\mathsf{pk}}(s)$.

Random oracle model

Let us model the hash function by an oracle $\mathcal{O}_{\mathsf{pk}}(\cdot)$ that evaluates $h \xleftarrow[u]{} \mathcal{H}_{\mathrm{all}}$ where $\mathcal{H}_{\mathrm{all}} = \{h : \mathcal{M} \to \mathcal{M}_{\mathsf{pk}}\}$ is a set of all functions.

Thus, the advantage is now computed as average

$$\mathsf{Adv}^{\mathsf{forge}}_{\mathcal{H}_{\mathrm{all}}}(\mathcal{A}) = \frac{1}{|\mathcal{H}_{\mathrm{all}}|} \cdot \sum_{h \in \mathcal{H}_{\mathrm{all}}} \mathsf{Adv}^{\mathsf{forge}}_{h}(\mathcal{A}) \enspace .$$

In reality, we substitute $\mathcal{H}_{\mathrm{all}}$ with a suitable hash family $\mathcal{H}$ and hope

$$\mathsf{Adv}^{\mathsf{forge}}_{\mathcal{H}}(\mathcal{A}) = \frac{1}{|\mathcal{H}|} \cdot \sum_{h \in \mathcal{H}} \mathsf{Adv}^{\mathsf{forge}}_{h}(\mathcal{A}) \approx \mathsf{Adv}^{\mathsf{forge}}_{\mathcal{H}_{\mathrm{all}}}(\mathcal{A})$$

for all relevant adversaries $\mathcal{A} \in \mathfrak{A}$ humans devise.

Security in the random oracle model

Theorem. Let $\mathcal{F}_{\mathrm{tp}}$ is (t, ε)-secure collection of trapdoor permutations such that $\mathrm{Map}_{\mathsf{pk}}(\cdot)$ is always τ-time computable. Then the FDH signature scheme is $(t - (q_h + q_s + 1) \cdot \tau, q_s, (q_h + q_s + 1) \cdot \varepsilon)$-secure against one more signature attack provided that the adversary can do up to q_h hash queries.

Drawbacks

- ▷ The result holds only in random oracle model.
- ▷ The security bound increases linearly with q_h and d_s.
- ▷ PSS signature scheme achieves a "sublinear" bound.
- ▷ The proof cannot be generalised to the standard model [Pallier2007].

The corresponding proof

- We can always assume that $\mathcal{A}$ queries hash for the forgery.
- The following evaluation strategies lead to identical results

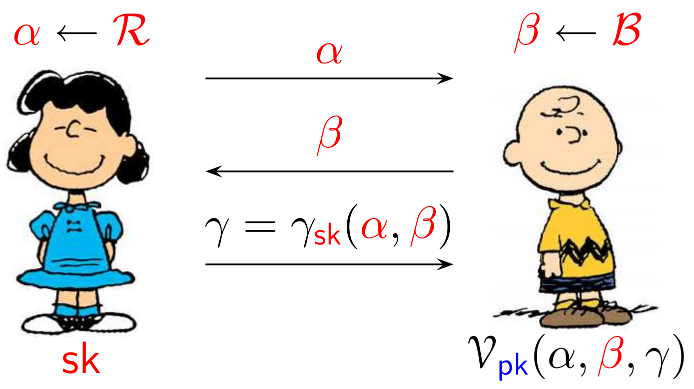
```
$\mathcal{O}_{pk}(m)$
    if $H[m] = \perp$ then
        $H[m] \xleftarrow{u} \mathcal{M}_{pk}$
    return $H[m]$
```

```
$\mathcal{O}_{pk}(m)$
    if $H[m] = \perp$ then
        $S[m] \xleftarrow{u} \mathcal{M}_{pk}$
        $H[m] \leftarrow \mathrm{Map}_{pk}(S[m])$
    return $H[m]$
```

- Let ε_i be the probability that forgery was successful and the ith hash query corresponded to the message.
- Then $\varepsilon_i \leq \varepsilon$, since otherwise we would have a too efficient inverter.

Digital Signatures Based on Proofs of Knowledge

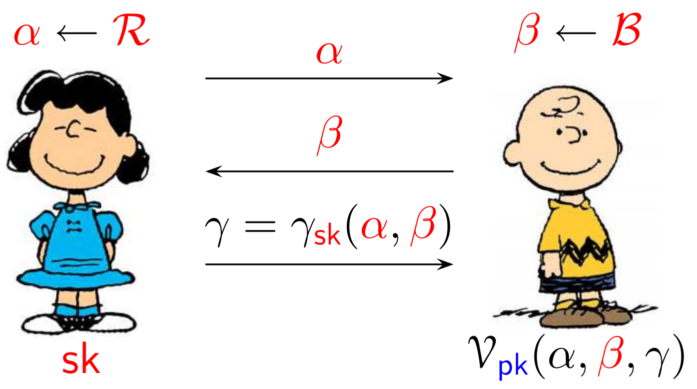
Sigma protocols


$\alpha \leftarrow \mathcal{R}$
$\beta \leftarrow \mathcal{B}$
α
β
$\gamma = \gamma_{\mathsf{sk}}(\alpha, \beta)$
sk
$\mathcal{V}_{\mathsf{pk}}(\alpha, \beta, \gamma)$

A *sigma protocol* for an efficiently computable relation $R \subseteq \{0,1\}^* \times \{0,1\}^*$ is a three move protocol that satisfies the following properties.

- ▷ **Σ-structure.** A prover first sends a commitment, next a verifier sends *varying* challenge, and then the prover must give a consistent response.
- ▷ **Functionality.** The protocol run between an honest prover $\mathcal{P}(\mathsf{sk})$ and verifier $\mathcal{V}(\mathsf{pk})$ is always accepting if $(\mathsf{sk}, \mathsf{pk}) \in R$.

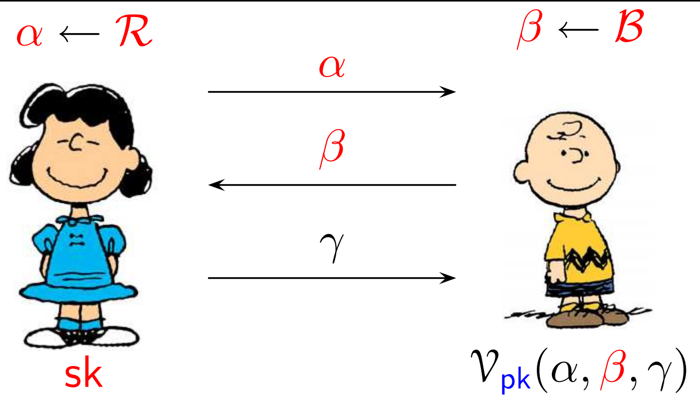
Security properties of sigma protocols


$\alpha \leftarrow \mathcal{R}$
$\beta \leftarrow \mathcal{B}$
α
β
$\gamma = \gamma_{\mathsf{sk}}(\alpha, \beta)$
sk
$\mathcal{V}_{\mathsf{pk}}(\alpha, \beta, \gamma)$

- **Perfect simulatability.** There exists an efficient *non-rewinding* simulator $\mathcal{S}$ such that the output distribution of a semi-honest verifier $\mathcal{V}_*$ in the real world and the output distribution of $\mathcal{S}^{\mathcal{V}_*}$ in the ideal world coincide.
- **Special soundness.** There exists an efficient extraction algorithm Extr that, given two accepting protocol runs $(\alpha, \beta_0, \gamma_0)$ and $(\alpha, \beta_1, \gamma_1)$ with $\beta_0 \neq \beta_1$ that correspond to pk, outputs sk_* such that $(\mathsf{sk}_*, \mathsf{pk}) \in R$

Fiat-Shamir heuristics

Fiat-Shamir heuristics converts any sigma-protocol to a signature scheme.


$\alpha \leftarrow \mathcal{R}$
$\beta \leftarrow \mathcal{B}$
α
β
γ
sk
$\mathcal{V}_{\mathsf{pk}}(\alpha, \beta, \gamma)$

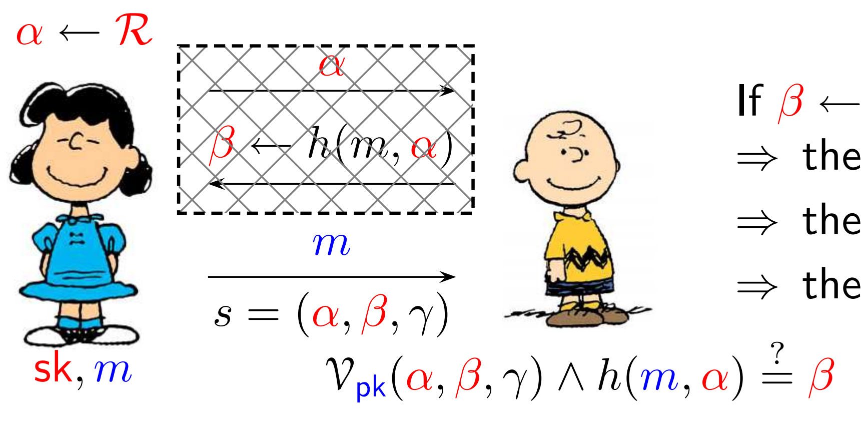
Sigma protocols are

$\Rightarrow$ interactive

$\Rightarrow$ non-transferrable

and cannot be linked to

$\Rightarrow$ particular messages


$\alpha \leftarrow \mathcal{R}$
α
$\beta \leftarrow h(m, \alpha)$
m
$s = (\alpha, \beta, \gamma)$
sk, m
$\mathcal{V}_{\mathsf{pk}}(\alpha, \beta, \gamma) \wedge h(m, \alpha) \stackrel{?}{=} \beta$

If $\beta \leftarrow h(m, \alpha)$ then

$\Rightarrow$ the signer cannot cheat

$\Rightarrow$ the protocol is non-interactive

$\Rightarrow$ the protocol is transferable

Generic signature schemes

Let π be a sigma protocol specified by the prover and verifier algorithms $\mathcal{P}_{\mathsf{sk}}$ and $\mathcal{V}_{\mathsf{pk}}$. Then the corresponding *generic signature scheme* is following.

Setup

Use output $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ of the sigma protocol setup.

Signing

To sign a message m, output (α, β, γ) as a signature where

$$\alpha \leftarrow \mathcal{P}_{\mathsf{sk}}, \quad \beta \leftarrow h(m, \alpha), \quad \gamma \leftarrow \mathcal{P}_{\mathsf{sk}}(\beta) \ .$$

Verification

A tuple $(m, \alpha, \beta, \gamma)$ is a valid signature if

$$\beta = h(m, \alpha) \wedge \mathsf{Ver}_{\mathsf{pk}}(\alpha, \beta, \gamma) = 1 \ .$$

Famous examples

Schnorr authentication protocol

◇ Schnorr signature scheme
◇ DSA algorithm
◇ ElGamal signature scheme
◇ Nyberg-Rueppel signature scheme

Fiat-Shamir identification protocol

◇ Feige-Fiat-Shamir signature scheme

Guillou-Quisquater identification protocol

◇ Guillou-Quisquater signature scheme

Security in the random oracle model

Basic Forking Lemma. Let κ be the knowledge error of the sigma protocol. If for a particular public key pk, a t-time adversary $\mathcal{A}$ manages to output valid signature by making at most ℓ queries to the random oracle and no queries to the signing oracle with probability ε, then there exist an extraction algorithm $\mathcal{K}$ for that runs in expected time

$$\mathbf{E}[\tau] = \Theta\left(\frac{(\ell + 2) \cdot t}{\varepsilon - \kappa}\right)$$

and returns the corresponding secret key sk.

Simulatability Lemma. Oracle calls to the signing oracle can be replaced with the runs of the sigma protocol simulator. The probability of simulation failures due to the contradicting assignments for $\mathcal{O}(\cdot)$ are negligible.

An encoding of the extraction task

Assume that $\mathcal{A}$ never queries the same value $h(m_i, \alpha_i)$ twice and that $\mathcal{A}$ itself verifies the validity of the candidate signature (m_{n+1}, s_{n+1}).

Let ω_0 denote the randomness used by $\mathcal{A}$ and let $\omega_1, \ldots \omega_{\ell+1}$ be the replies for the hash queries $h(m_i, \alpha_i)$. Now define

$$A(\omega_0, \omega_1, \ldots, \omega_{\ell+1}) = \begin{cases} i, & \text{if the } i^{\text{th}} \text{ reply } \omega_i \text{ is used in forgery} \enspace, \\ 0, & \text{if } \mathcal{A} \text{ fails} \enspace . \end{cases}$$

- ▷ For any $\overline{\boldsymbol{\omega}} = (\omega_0, \ldots, \omega_{i-1}, \overline{\omega}_i, \ldots, \overline{\omega}_{\ell+1})$, $\mathcal{A}$ behaves identically up to the i^{th} query as with the randomness $\boldsymbol{\omega}$.
- ▷ To extract the secret key sk, we must find $\boldsymbol{\omega}$ and $\overline{\boldsymbol{\omega}}$ such that $A(\boldsymbol{\omega}) = i$ and $A(\overline{\boldsymbol{\omega}}) = i$ and $\omega_i \neq \overline{\omega}_i$.

Classical algorithm

Rewind:

1. Probe random entries $A(\boldsymbol{\omega})$ until $A(r, c) \neq 0$.
2. Store the matrix location $\boldsymbol{\omega}$ and the rewinding point $i \leftarrow A(\boldsymbol{\omega})$.
3. Probe random entries $A(\overline{\boldsymbol{\omega}})$ until $A(\overline{\boldsymbol{\omega}}) = i$.
4. Output the location tuple $(\boldsymbol{\omega}, \overline{\boldsymbol{\omega}})$.

Rewind-Exp:

1. Repeat the procedure Rewind until $\omega_i \neq \overline{\omega}_i$.
2. Use the knowledge extraction lemma to extract sk.

Average-case running time

Theorem. If a array $A(\boldsymbol{\omega})$ with entries in $\{0,\ldots,\ell\}$ contains ε-fraction of nonzero entries, then Rewind and Rewind-Exp make on average

$$\mathbf{E}[\text{probes}|\text{Rewind}] = \frac{2}{\varepsilon}$$
$$\mathbf{E}[\text{probes}|\text{Rewind-Exp}] = \frac{\ell+1}{\varepsilon-\kappa}$$

probes where the *knowledge error*

$$\kappa = \sum_{i=1}^{\ell} \Pr\left[\omega_i = \overline{\omega}_i\right] \quad .$$

Proof. We prove this theorem in another lecture.

Interpretation of the Results

Objective and subjective security

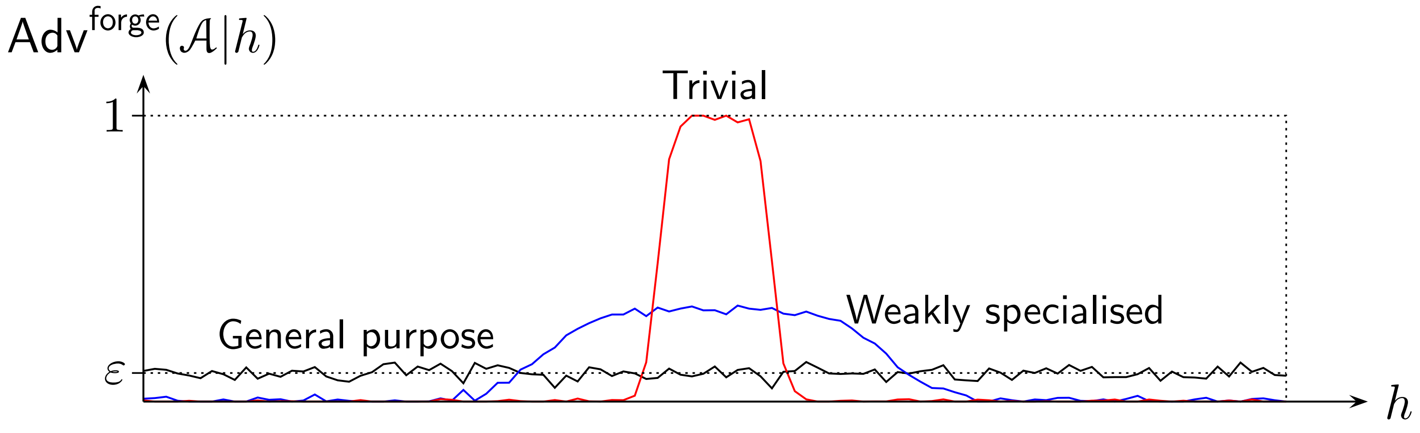
All forms of the Forking Lemma are stated with respect to a fixed key and thus give only soundness guarantees.

If we believe that no human can devise an algorithm that for a *particular* public key pk computes the corresponding secret key, then we also get the corresponding *subjective* security guarantee.

For objective security guarantees, we have to consider average success probability over all public keys. In this situation, the knowledge extraction with high probability is overkill. Fixed number of probes together with Jensen's inequality give more tight bounds.

All these security guarantees hold in the random oracle model!

Average case nature of advantages



The limit on the average advantage over all functions means:

- ▷ An attack algorithm $\mathcal{A}$ can be successful on few functions
- ▷ For randomly chosen function family $\mathcal{H}$ the corresponding average advantage is comparable with high probability over the choice of $\mathcal{H}$.

Such argumentation does not rule out possibility that one can choose adaptively a specialised attack algorithm $\mathcal{A}$ based on the description of h.

Security against generic attacks

An adaptive choice of a specialised attack algorithm implies that the attack depends on the description of the hash function and not the family $\mathcal{H}$.

Often, it is advantageous to consider only generic attacks that depend on the description of function family $\mathcal{H}$ and use only black-box access to the function h. Therefore, we can consider two oracles $\mathcal{O}_{\mathcal{H}_{\mathrm{all}}}$ and $\mathcal{O}_{\mathcal{H}}$.

If $\mathcal{H}$ is pseudorandom function family then for any generic attack, we can substitute $\mathcal{H}$ with the $\mathcal{H}_{\mathrm{all}}$ and the success decreases marginally.

Theorem. Security in the random oracle model implies security against generic attacks if $\mathcal{H}$ is a pseudorandom function family.

- ▷ The assumption that attackers use only generic attacks is subjective.
- ▷ Such an assumption are not universal, i.e., there are settings where this assumption is clearly irrational (various non-instantiability results).

Why the random oracle argument truly fails?

All currently used hash functions are iterative

$$h(m_1, \ldots, m_k) = f(\cdots f(f(\mathsf{iv}, m_1), m_2), \ldots, m_k) \ .$$

Consequently, the class of generic attack $\mathfrak{A}_{\mathrm{bb}}$ that treats $h(\cdot)$ as a black box without internal structure is sub-optimal.

Instead, we should consider a wider class of attacks that treat f as a black box, but still try to exploit iterative structure of the hash function.

However, the corresponding black box adversaries $\mathcal{A}^{f(\cdot),h(\cdot)}$ can distinguish $\mathcal{H}$ form $\mathcal{H}_{\mathrm{all}}$ with high probability.

As a result, there might be a successful attack strategy $\mathcal{A}^{f(\cdot),h(\cdot)}$ that works for all possible iterative hash functions, although the signature scheme is secure in the random oracle model.