Commitment Schemes

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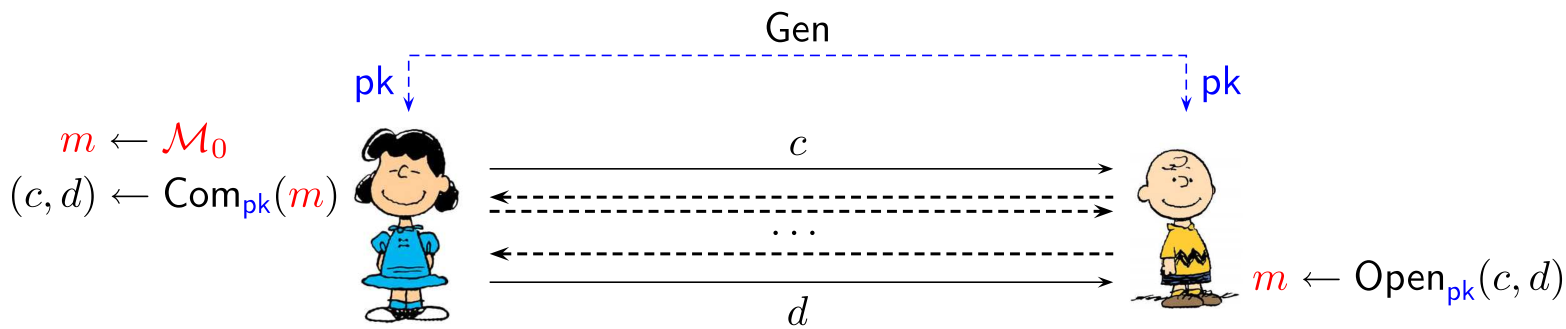
Formal Syntax

Canonical use case



- ▷ A randomised key generation algorithm Gen outputs a public parameters pk that must be authentically transferred all participants.
- ▷ A commitment function $\mathsf{Com}_{\mathsf{pk}} : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ takes in a plaintext and outputs a corresponding digest c and decommitment string d.
- ▷ A commitment can be opened with $\mathsf{Open}_{\mathsf{pk}} : \mathcal{C} \times \mathcal{D} \to \mathcal{M} \cup \{\perp\}$.
- ▷ The commitment primitive is functional if for all pk $\leftarrow$ Gen and $m \in \mathcal{M}$:

$$\mathsf{Open}_{\mathsf{pk}}(\mathsf{Com}_{\mathsf{pk}}(m)) = m \enspace .$$

Binding property

A commitment scheme is (t, ε)-binding if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{bind}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon \enspace ,$$

where

$$\mathcal{G}^{\mathcal{A}}$$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (c, d_0, d_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ m_0 \leftarrow \mathsf{Open}_{\mathsf{pk}}(c, d_0) \\ m_1 \leftarrow \mathsf{Open}_{\mathsf{pk}}(c, d_1) \\ \text{if } m_0 = \perp \text{ or } m_1 = \perp \text{ then return } 0 \\ \text{else return } \neg[m_0 \stackrel{?}{=} m_1] \end{array}\right.$$

Collision resistant hash functions

A function family $\mathcal{H}$ is (t, ε)-collision resistant if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{cr}}_{\mathcal{H}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon \ ,$$

where

$$\begin{array}{l}\mathcal{G}^{\mathcal{A}} \\ \left[\begin{array}{l} h \xleftarrow[u]{} \mathcal{H} \\ (m_0, m_1) \leftarrow \mathcal{A}(h) \\ \text{if } m_0 = m_1 \text{ then return } 0 \\ \text{else return } [h(m_0) \stackrel{?}{=} h(m_1)] \end{array}\right.\end{array}$$

Hash commitments

Let $\mathcal{H}$ be (t, ε)-collision resistant hash function family. Then we can construct a binding commitment:

- ▷ The setup algorithm returns $h \xleftarrow[u]{} \mathcal{H}$ as a public parameter.
- ▷ To commit m, return $h(m)$ as digest and m as a decommitment string.
- ▷ The message m is a valid opening of c if $h(m) = c$.

Usage

- ▷ Integrity check for files and file systems in general.
- ▷ Minimisation of memory footprint in servers:
  1. A server stores the hash $c \leftarrow h(m)$ of an initial application data m.
  2. Data is stored by potentially malicious clients.
  3. Provided data m' is correct if $h(m') = c$.

Hiding property

A commitment scheme is (t, ε)-hiding if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{hid}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \ ,$$

where

$$\mathcal{G}_0^{\mathcal{A}} \qquad\qquad\qquad\qquad \mathcal{G}_1^{\mathcal{A}}$$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ (c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m_0) \\ \text{return } \mathcal{A}(c) \end{array}\right. \qquad \left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ (c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m_1) \\ \text{return } \mathcal{A}(c) \end{array}\right.$$

Any cryptosystem is a commitment scheme

Setup:

Compute $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ and delete sk and output pk.

Commitment:

To commit m, sample necessary randomness $r \leftarrow \mathcal{R}$ and output:

$$\begin{cases} c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m; r) \enspace , \\ d \leftarrow (m, r) \enspace . \end{cases}$$

Opening:

A tuple (c, m, r) is a valid decommitment if $c = \mathsf{Enc}_{\mathsf{pk}}(m; r)$.

Security guarantees

If a cryptosystem is (t, ε)-IND-CPA secure and functional, then the resulting commitment scheme is (t, ε)-hiding and perfectly binding.

- ◇ We can construct commitment schemes from the ElGamal and Goldwasser-Micali cryptosystems.
- ◇ For the ElGamal cryptosystem, one can create public parameters pk without the knowledge of the secret key sk.
- ◇ The knowledge of the secret key sk allows a participant to extract messages from the commitments.
- ◇ The extractability property is useful in security proofs.

Dedicated Commitment Schemes

Modified Naor commitment scheme

Setup:

Choose a random n-bit string $\mathsf{pk} \xleftarrow[u]{} \{0,1\}^n$.
Let $f : \{0,1\}^k \to \{0,1\}^n$ be a pseudorandom generator.

Commitment:

To commit $m \in \{0,1\}$, generate $d \leftarrow \{0,1\}^k$ and compute digest

$$c \leftarrow \begin{cases} f(d), & \text{if } m = 0 \enspace, \\ f(d) \oplus \mathsf{pk}, & \text{if } m = 1 \enspace. \end{cases}$$

Opening:

Given (c, d) check whether $c = f(d)$ or $c = f(d) \oplus \mathsf{pk}$.

Security guarantees

If $f : \{0,1\}^k \rightarrow \{0,1\}^n$ is (t,ε)-secure pseudorandom generator, then the modified Naor commitment scheme is $(t,2\varepsilon)$-hiding and 2^{2k-n}-binding.

Proof

Hiding claim is obvious, since we can change $f(d)$ with uniform distribution. For the binding bound note that

$$|\mathcal{PK}_{\mathrm{bad}}| = \#\left\{\mathsf{pk} : \exists d_0, d_1 :\ f(d_0) \oplus f(d_1) = \mathsf{pk}\right\} \leq 2^{2k}$$
$$|\mathcal{PK}_{\mathrm{all}}| = \#\left\{0,1\right\}^n = 2^n$$

and thus

$$\mathsf{Adv}^{\mathsf{bind}}(\mathcal{A}) \leq \Pr\left[\mathsf{pk} \in \mathcal{PK}_{\mathrm{bad}}\right] \leq 2^{2k-n} \quad .$$

Discrete logarithm

Let $\mathbb{G} = \langle g \rangle$ be a q-element group that is generated by a single element g. Then for any $y \in \mathbb{G}$ there exists a minimal value $0 \leq x \leq q$ such that

$$g^x = y \quad \Leftrightarrow \quad x = \log_g y \ .$$

A group $\mathbb{G}$ is (t, ε)-secure DL group if for any t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\mathsf{dl}}_{\mathbb{G}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon \ ,$$

where

$$\begin{array}{l} \mathcal{G}^{\mathcal{A}} \\ \left[\begin{array}{l} y \xleftarrow[u]{} \mathbb{G} \\ x \leftarrow \mathcal{A}(y) \\ \text{return } [g^x \stackrel{?}{=} y] \end{array}\right. \end{array}$$

Pedersen commitment scheme

Setup:

Let q be a prime and let $\mathbb{G} = \langle g \rangle$ be a q-element DL-group.
Choose y uniformly from $\mathbb{G} \setminus \{1\}$ and set $\mathsf{pk} \leftarrow (g, y)$.

Commitment:

To commit $m \in \mathbb{Z}_q$, choose $r \xleftarrow[u]{} \mathbb{Z}_q$ and output

$$\begin{cases} c \leftarrow g^m y^r & , \\ d \leftarrow (m, r) & . \end{cases}$$

Opening:

A tuple (c, m, r) is a valid decommitment if $c = g^m y^r$.

Security guarantees

Assume that $\mathbb{G}$ is (t, ε)-secure discrete logarithm group. Then the Pedersen commitment is perfectly hiding and (t, ε)-binding commitment scheme.

Proof

- ▷ HIDING. The factor y^r has uniform distribution over $\mathbb{G}$, since $y^r = g^{xr}$ for $x \neq 0$ and $\mathbb{Z}_q$ is simple ring: $x \cdot \mathbb{Z}_q = \mathbb{Z}_q$.
- ▷ BINDING. A valid double opening reveals a discrete logarithm of y:

$$g^{m_0} y^{r_0} = g^{m_1} y^{r_1} \quad \Leftrightarrow \quad \log_g y = \frac{m_1 - m_0}{r_0 - r_1} \enspace .$$

Note that $r_0 \neq r_1$ for valid double opening. Hence, a double opener $\mathcal{A}$ can be converted to a solver of discrete logarithm.

Other Useful Properties

Extractability

A commitment scheme is (t, ε)-extractable if there exists a modified setup procedure $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^*$ such that

- ▷ the distribution of public parameters pk coincides with the original setup;
- ▷ there exists an efficient extraction function $\mathsf{Extr}_{\mathsf{sk}} : \mathcal{C} \rightarrow \mathcal{M}$ such that for any t-time adversary $\mathsf{Adv}^{\mathsf{ext}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon$, where

$$\mathcal{G}^{\mathcal{A}}$$

$$\left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^* \\ (c, d) \leftarrow \mathcal{A}(\mathsf{pk}) \\ \text{if } \mathsf{Open}_{\mathsf{pk}}(c, d) = \perp \text{ then return } 0 \\ \text{else return } \neg[\mathsf{Open}_{\mathsf{pk}}(c, d) \stackrel{?}{=} \mathsf{Extr}_{\mathsf{sk}}(c)] \end{array}\right.$$

Equivocability

A commitment scheme is equivocable if there exists

- ▷ a modified setup procedure $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^*$
- ▷ a modified fake commitment procedure $(\hat{c}, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}}$
- ▷ an efficient equivocation function $\hat{d} \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(\hat{c}, \sigma, m)$

such that

- ▷ the distribution of public parameters pk coincides with the original setup;
- ▷ fake commitments $\hat{c}$ are indistinguishable from real commitments
- ▷ fake commitments $\hat{c}$ can be opened to arbitrary values

$$\forall m \in \mathcal{M}, (\hat{c}, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}}, \hat{d} \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(\hat{c}, \sigma, m) : \mathsf{Open}_{\mathsf{pk}}(\hat{c}, \hat{d}) \equiv m \enspace .$$

- ▷ opening fake and real commitments are indistinguishable.

Formal security definition

A commitment scheme is (t, ε)-equivocable if for any t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\mathsf{eqv}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \enspace ,$$

where

$$\mathcal{G}_0^{\mathcal{A}}\qquad\qquad\qquad\qquad \mathcal{G}_1^{\mathcal{A}}$$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ \mathsf{repeat} \\ \quad \left| \begin{array}{l} m_i \leftarrow \mathcal{A} \\ (c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m) \\ \mathcal{A}(c, d) \end{array} \right. \\ \mathsf{until}\ m_i = \perp \\ \mathsf{return}\ \mathcal{A} \end{array}\right. \qquad \left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^* \\ \mathsf{repeat} \\ \quad \left| \begin{array}{l} m_i \leftarrow \mathcal{A}, (c, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}} \\ d \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(c, \sigma, m) \\ \mathcal{A}(c, d) \end{array} \right. \\ \mathsf{until}\ m_i = \perp \\ \mathsf{return}\ \mathcal{A} \end{array}\right.$$

A famous example

The Pedersen is perfectly equivocable commitment.

▷ **Setup**. Generate $x \leftarrow \mathbb{Z}_q^*$ and set $y \leftarrow g^x$.

▷ **Fake commitment**. Generate $s \leftarrow \mathbb{Z}_q$ and output $\hat{c} \leftarrow g^s$.

▷ **Equivocation.** To open $\hat{c}$, compute $r \leftarrow (s - m) \cdot x^{-1}$.

Proof

▷ Commitment value c has uniform distribution.

▷ For fixed c and m, there exists a unique value of r.

Equivocation leads to perfect simulation of (c, d) pairs.

Homomorphic commitments

A commitment scheme is $\otimes$-homomorphic if there exists an efficient coordinate-wise multiplication operation $\cdot$ defined over $\mathcal{C}$ and $\mathcal{D}$ such that

$$\mathsf{Com}_{\mathsf{pk}}(m_1) \cdot \mathsf{Com}_{\mathsf{pk}}(m_2) \equiv \mathsf{Com}_{\mathsf{pk}}(m_1 \otimes m_2) \enspace ,$$

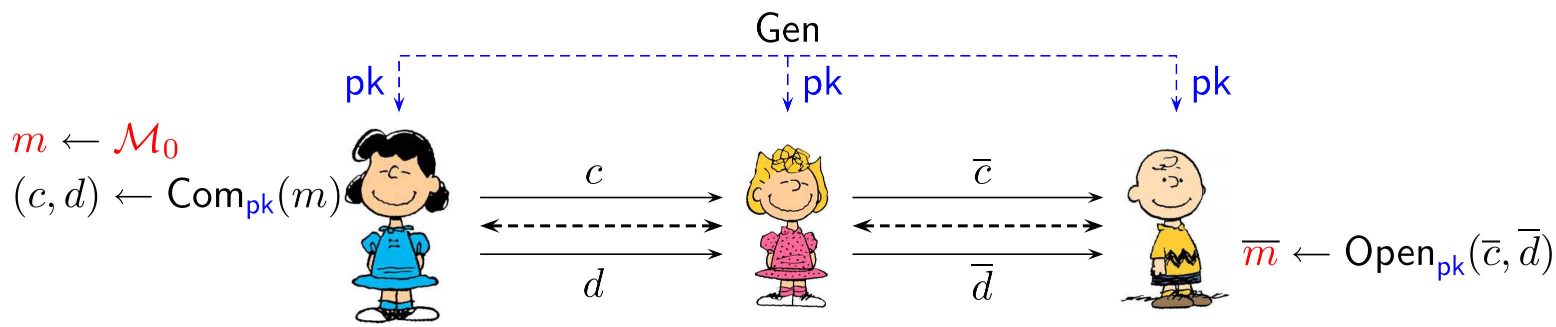
where the distributions coincide even if $\mathsf{Com}_{\mathsf{pk}}(m_1)$ is fixed.

Examples

- ElGamal commitment scheme
- Pedersen commitment scheme

Active Attacks

Non-malleability wrt opening



A commitment scheme is non-malleable wrt. opening if an adversary

▷ who knows the input distribution $\mathcal{M}_0$

cannot alter commitment and decommitment values c, d on the fly

▷ so that the opening $\overline{m}$ that is related to original message m.

Commitment c does not help the adversary to create other commitments.

Formal definition

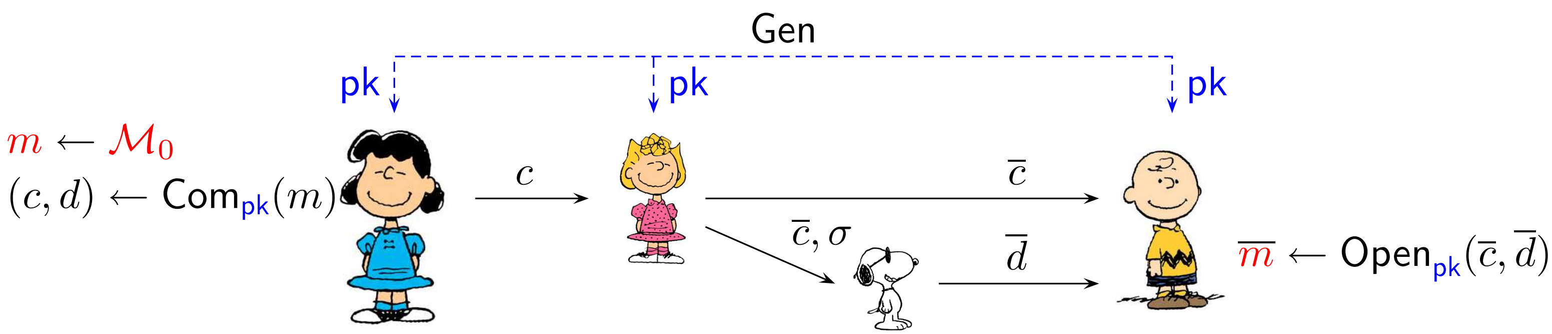
$\mathcal{G}_0^{\mathcal{A}}$

$$
\left[\begin{array}{l}
\mathsf{pk} \leftarrow \mathsf{Gen} \\
\mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\
m \leftarrow \mathcal{M}_0 \\
(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m) \\
\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n \leftarrow \mathcal{A}(c) \\
\hat{d}_1, \ldots \hat{d}_n \leftarrow \mathcal{A}(d) \\
\text{if } c \in \{\hat{c}_1, \ldots, \hat{c}_n\} \text{ then return } 0 \\
\hat{m}_i \leftarrow \mathsf{Open}_{\mathsf{pk}}(\hat{c}_i, \hat{d}_i),\ i = 1, \ldots, n \\
\text{return } \pi(m, \hat{m}_1, \ldots, \hat{m}_n)
\end{array}\right.
$$

$\mathcal{G}_1^{\mathcal{A}}$

$$
\left[\begin{array}{l}
\mathsf{pk} \leftarrow \mathsf{Gen} \\
\mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\
m \leftarrow \mathcal{M}_0,\ \overline{m} \leftarrow \mathcal{M}_0 \\
(\overline{c}, \overline{d}) \leftarrow \mathsf{Com}_{\mathsf{pk}}(\overline{m}) \\
\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n \leftarrow \mathcal{A}(\overline{c}) \\
\hat{d}_1, \ldots \hat{d}_n \leftarrow \mathcal{A}(\overline{d}) \\
\text{if } c \in \{\hat{c}_1, \ldots, \hat{c}_n\} \text{ then return } 0 \\
\hat{m}_i \leftarrow \mathsf{Open}_{\mathsf{pk}}(\hat{c}_i, \hat{d}_i),\ i = 1, \ldots, n \\
\text{return } \pi(m, \hat{m}_1, \ldots, \hat{m}_n)
\end{array}\right.
$$

Non-malleability wrt commitment



A commitment scheme is non-malleable wrt. opening if an adversary $\mathcal{A}_1$

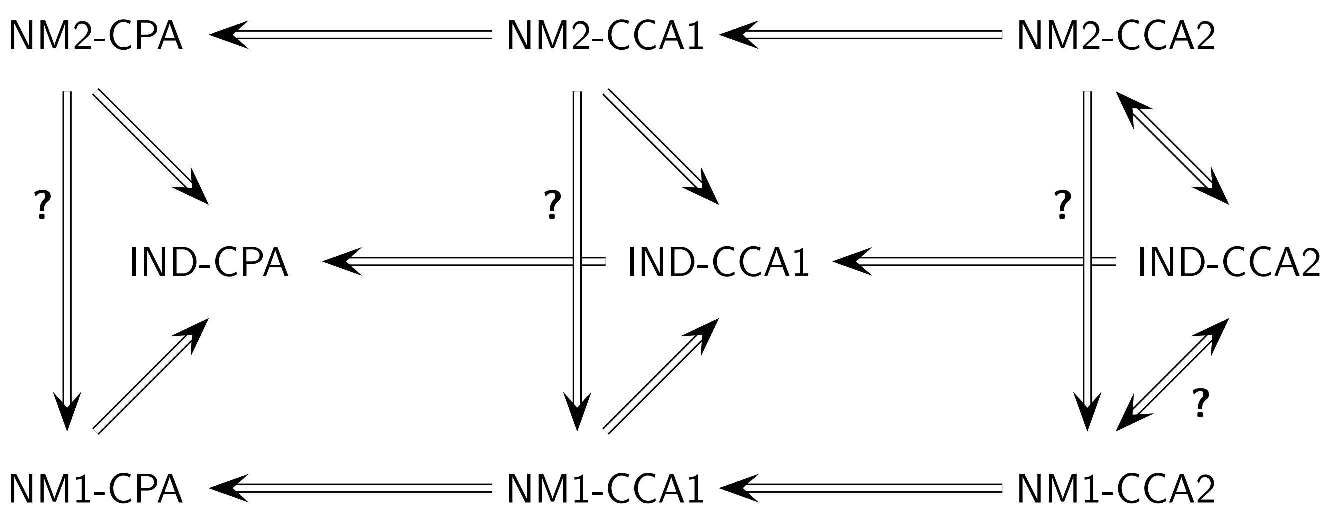
- ▷ who knows the input distribution $\mathcal{M}_0$

cannot alter the commitment value c on the fly

- ▷ so that an unbounded adversary $\mathcal{A}_2$ cannot open the altered commitment value $\overline{c}$ to a message $\overline{m}$ that is related to original message m.

Commitment c does not help the adversary to create other commitments even if some secret values are leaked after the creation of c and $\overline{c}$.

Homological classification



Can we define decommitment oracles such that the graph depicted above captures relations between various notions where

- ▷ NM1-XXX denotes non-malleability wrt opening,
- ▷ NM2-XXX denotes non-malleability wrt commitment.

Coin flipping

Coin flipping by telephone

Gen

$b_1 \xleftarrow{u} \{0,1\}$	pk		pk	$b_2 \xleftarrow{u} \{0,1\}$
$(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(b_1)$		$\xrightarrow{\quad c \quad}$		
		$\xleftarrow{\quad b_2 \quad}$		
		$\xrightarrow{\quad d \quad}$		$b_1 \leftarrow \mathsf{Open}_{\mathsf{pk}}(c, d)$
return $b_1 \oplus b_2$				return $b_1 \oplus b_2$

The protocol above assures that participants output a uniformly distributed bit even if one of the participants is malicious.

- ▷ If the commitment scheme is perfectly binding, then Lucy can also generate public parameters for the commitment scheme.
- ▷ If the commitment scheme is perfectly hiding, then Charlie can also generate public parameters for the commitment scheme.

Weak security guarantee

Theorem. If we consider only such adversarial strategies that do not cause premature halting and additionally assume that the commitment scheme is (t, ε_1)-hiding and (t, ε_2)-binding, then

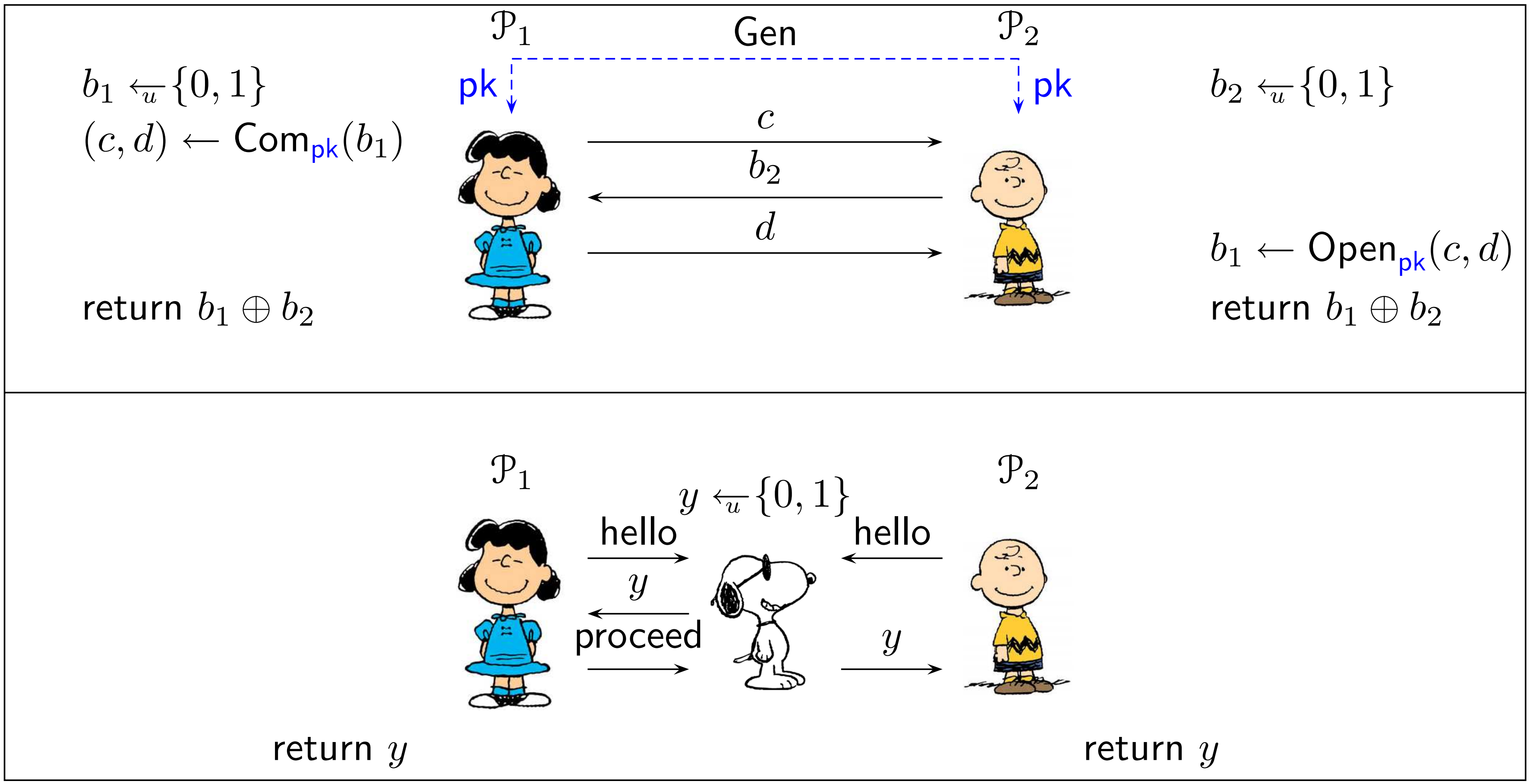
$$\frac{1}{2} - \max\{\varepsilon_1, \varepsilon_2\} \leq \Pr[b_1 \oplus b_2 = 1] \leq \frac{1}{2} + \max\{\varepsilon_1, \varepsilon_2\}$$

provided that at least one participant is honest.

Proof

- ▷ Lucy cannot cheat unless it double opens the commitment.
- ▷ As commitment is hiding the Charlie cannot guess b_1.

Real versus ideal world approach



Strong security guarantee

Theorem. If a commitment scheme is $(k \cdot t, \varepsilon_1)$-hiding and (t, ε_2)-binding, then for any plausible t-time real world adversary there exists $\mathrm{O}(k \cdot t)$-time ideal world adversary such that the output distributions in the real and ideal world are $\max\left\{2^{-k} + 2k \cdot \varepsilon_1, \varepsilon_2\right\}$-close.