MTAT.07.003 Cryptology II

Commitment Schemes

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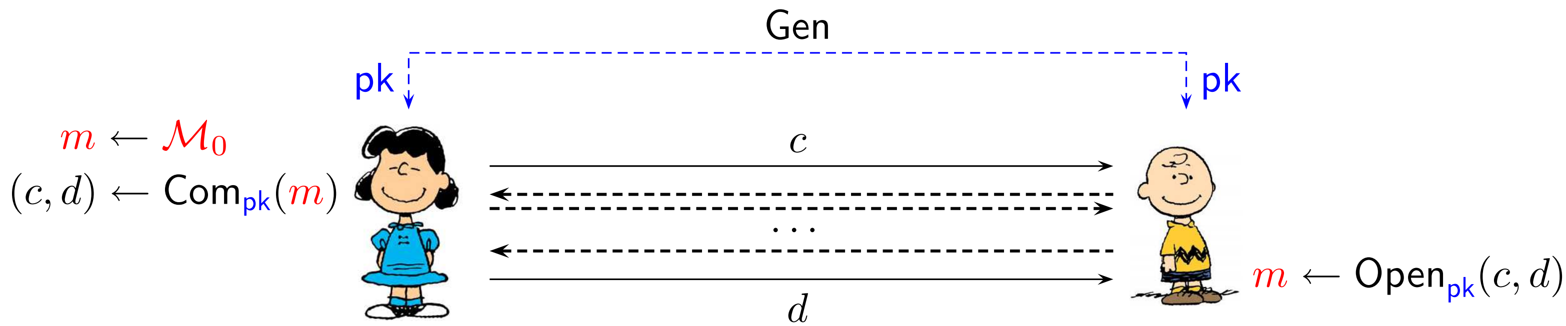
Formal Syntax

Canonical use case



- ▷ A randomised key generation algorithm Gen outputs a *public parameters* pk that must be authentically transferred all participants.
- ▷ A commitment function $\mathsf{Com}_{\mathsf{pk}} : \mathcal{M} \to \mathcal{C} \times \mathcal{D}$ takes in a *plaintext* and outputs a corresponding *digest* c and decommitment string d.
- ▷ A commitment can be opened with $\mathsf{Open}_{\mathsf{pk}} : \mathcal{C} \times \mathcal{D} \to \mathcal{M} \cup \{\perp\}$.
- ▷ The commitment primitive is *functional* if for all $\mathsf{pk} \leftarrow \mathsf{Gen}$ and $m \in \mathcal{M}$:

$$\mathsf{Open}_{\mathsf{pk}}(\mathsf{Com}_{\mathsf{pk}}(m)) = m \ .$$

Binding property

A commitment scheme is (t, ε)-*binding* if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{bind}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \le \varepsilon \ ,$$

where the challenge game is following

$$\mathcal{G}^{\mathcal{A}}$$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (c, d_0, d_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ m_i \leftarrow \mathsf{Open}_{\mathsf{pk}}(c, d_i) \textbf{for } i = 0, 1 \\ \text{if } m_0 = \bot \text{ or } m_1 = \bot \text{ then } \textbf{return } 0 \\ \text{else } \textbf{return } \neg[m_0 \stackrel{?}{=} m_1] \end{array}\right.$$

Collision resistant hash functions

A function family $\mathcal{H}$ is (t, ε)-*collision resistant* if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{cr}}_{\mathcal{H}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon \enspace ,$$

where the challenge game is following

$$\begin{array}{l}\mathcal{G}^{\mathcal{A}}\\ \left[\begin{array}{l} h \xleftarrow[u]{} \mathcal{H} \\ (m_0, m_1) \leftarrow \mathcal{A}(h) \\ \text{if } m_0 = m_1 \text{ then } \textbf{return } 0 \\ \text{else } \textbf{return } [h(m_0) \overset{?}{=} h(m_1)] \end{array}\right.\end{array}$$

Hash commitments

Let $\mathcal{H}$ be (t, ε)-collision resistant hash function family. Then we can construct a binding commitment:

- ▷ The setup algorithm returns $h \xleftarrow[u]{} \mathcal{H}$ as a public parameter.
- ▷ To commit m, return $h(m)$ as digest and m as a decommitment string.
- ▷ The message m is a valid opening of c if $h(m) = c$.

Usage

- ▷ Integrity check for files and file systems in general.
- ▷ Minimisation of memory footprint in servers:
  1. A server stores the hash $c \leftarrow h(m)$ of an initial application data m.
  2. Data is stored by potentially malicious clients.
  3. Provided data m' is correct if $h(m') = c$.

Hiding property

A commitment scheme is (t,ε)-*hiding* if for any t-time adversary $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{hid}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \le \varepsilon \ ,$$

where

$$\mathcal{G}_0^{\mathcal{A}} \qquad\qquad\qquad\qquad \mathcal{G}_1^{\mathcal{A}}$$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ (c,d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m_0) \\ \textbf{return } \mathcal{A}(c) \end{array}\right. \qquad\qquad \left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ (c,d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m_1) \\ \textbf{return } \mathcal{A}(c) \end{array}\right.$$

Any cryptosystem is a commitment scheme

Setup:

Compute $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}$ and delete sk and output pk.

Commitment:

To commit m, sample necessary randomness $r \leftarrow \mathcal{R}$ and output:

$$\begin{cases} c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m; r) \enspace , \\ d \leftarrow (m, r) \enspace . \end{cases}$$

Opening:

A tuple (c, m, r) is a valid decommitment of m if $c = \mathsf{Enc}_{\mathsf{pk}}(m; r)$.

Security guarantees

If a cryptosystem is (t, ε)-IND-CPA secure and functional, then the resulting commitment scheme is (t, ε)-hiding and perfectly binding.

- ◇ We can construct commitment schemes from the ElGamal and Goldwasser-Micali cryptosystems.
- ◇ For the ElGamal cryptosystem, one can create public parameters pk without the knowledge of the secret key sk.
- ◇ The knowledge of the secret key sk allows a participant to extract messages from the commitments.
- ◇ The extractability property is useful in security proofs.

Simple Commitment Schemes

Modified Naor commitment scheme

Setup:

Choose a random n-bit string pk $\xleftarrow[u]{} \{0,1\}^n$.
Let $f : \{0,1\}^k \to \{0,1\}^n$ be a pseudorandom generator.

Commitment:

To commit $m \in \{0,1\}$, generate $d \leftarrow \{0,1\}^k$ and compute digest

$$c \leftarrow \begin{cases} f(d), & \text{if } m = 0 \ , \\ f(d) \oplus \mathsf{pk}, & \text{if } m = 1 \ . \end{cases}$$

Opening:

Given (c, d) check whether $c = f(d)$ or $c = f(d) \oplus \mathsf{pk}$.

Security guarantees

If $f : \{0,1\}^k \to \{0,1\}^n$ is (t, ε)-secure pseudorandom generator, then the modified Naor commitment scheme is $(t, 2\varepsilon)$-hiding and 2^{2k-n}-binding.

Proof

Hiding claim is obvious, since we can change $f(d)$ with uniform distribution. For the binding bound note that

$$\begin{aligned} |\mathcal{PK}_{\mathrm{bad}}| &= \#\left\{\mathsf{pk} : \exists d_0, d_1 : \ f(d_0) \oplus f(d_1) = \mathsf{pk}\right\} \leq 2^{2k} \\ |\mathcal{PK}_{\mathrm{all}}| &= \#\left\{0,1\right\}^n = 2^n \end{aligned}$$

and thus

$$\mathsf{Adv}^{\mathsf{bind}}(\mathcal{A}) \leq \Pr\left[\mathsf{pk} \in \mathcal{PK}_{\mathrm{bad}}\right] \leq 2^{2k-n} \quad .$$

Discrete logarithm

Let $\mathbb{G} = \langle g \rangle$ be a q-element group that is generated by a single element g. Then for any $y \in \mathbb{G}$ there exists a minimal value $0 \leq x \leq q$ such that

$$g^x = y \quad \Leftrightarrow \quad x = \log_g y \enspace .$$

A group $\mathbb{G}$ is *(t, ε)-secure DL group* if for any t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\mathsf{dl}}_{\mathbb{G}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon \enspace ,$$

where

$$\mathcal{G}^{\mathcal{A}}$$

$$\left[\begin{array}{l} y \xleftarrow[u]{} \mathbb{G} \\ x \leftarrow \mathcal{A}(y) \\ \textbf{return } [g^x \stackrel{?}{=} y] \end{array}\right.$$

Pedersen commitment scheme

Setup:

Let q be a prime and let $\mathbb{G} = \langle g \rangle$ be a q-element DL-group.
Choose y uniformly from $\mathbb{G} \setminus \{1\}$ and set pk $\leftarrow (g, y)$.

Commitment:

To commit $m \in \mathbb{Z}_q$, choose $r \xleftarrow[u]{} \mathbb{Z}_q$ and output

$$\begin{cases} c \leftarrow g^m y^r & , \\ d \leftarrow (m, r) & . \end{cases}$$

Opening:

A tuple (c, m, r) is a valid decommitment for m if $c = g^m y^r$.

Security guarantees

Assume that $\mathbb{G}$ is (t, ε)-secure discrete logarithm group. Then the Pedersen commitment is perfectly hiding and (t, ε)-binding commitment scheme.

Proof

- ▷ HIDING. The factor y^r has uniform distribution over $\mathbb{G}$, since $y^r = g^{xr}$ for $x \neq 0$ and $\mathbb{Z}_q$ is simple ring: $x \cdot \mathbb{Z}_q = \mathbb{Z}_q$.
- ▷ BINDING. A valid double opening reveals a discrete logarithm of y:

$$g^{m_0} y^{r_0} = g^{m_1} y^{r_1} \quad \Leftrightarrow \quad \log_g y = \frac{m_1 - m_0}{r_0 - r_1} \enspace .$$

Note that $r_0 \neq r_1$ for valid double opening. Hence, a double opener $\mathcal{A}$ can be converted to a discrete logarithm finder.

Other Useful Properties

Extractability

A commitment scheme is (t, ε)-*extractable* if there exists a modified setup procedure $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^*$ such that

- ▷ the distribution of public parameters pk coincides with the original setup;
- ▷ there exists an efficient extraction function $\mathsf{Extr}_{\mathsf{sk}} : \mathcal{C} \rightarrow \mathcal{M}$ such that for any t-time adversary $\mathsf{Adv}^{\mathsf{ext}}(\mathcal{A}) = \Pr\left[\mathcal{G}^{\mathcal{A}} = 1\right] \leq \varepsilon$ where

$$\mathcal{G}^{\mathcal{A}}$$

$$\left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^* \\ (c, d) \leftarrow \mathcal{A}(\mathsf{pk}) \\ \text{if } \mathsf{Open}_{\mathsf{pk}}(c, d) = \perp \text{ then } \textbf{return } 0 \\ \text{else } \textbf{return } \neg[\mathsf{Open}_{\mathsf{pk}}(c, d) \stackrel{?}{=} \mathsf{Extr}_{\mathsf{sk}}(c)] \end{array}\right.$$

Equivocability

A commitment scheme is *equivocable* if there exists

- ▷ a modified setup procedure $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^*$
- ▷ a modified fake commitment procedure $(\hat{c}, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}}$
- ▷ an efficient equivocation algorithm $\hat{d} \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(\hat{c}, \sigma, m)$

such that

- ▷ the distribution of public parameters pk coincides with the original setup;
- ▷ fake commitments $\hat{c}$ are indistinguishable from real commitments
- ▷ fake commitments $\hat{c}$ can be opened to arbitrary values

$$\forall m \in \mathcal{M}, (\hat{c}, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}}, \hat{d} \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(\hat{c}, \sigma, m) : \mathsf{Open}_{\mathsf{pk}}(\hat{c}, \hat{d}) \equiv m \enspace .$$

- ▷ opening fake and real commitments are indistinguishable.

Formal security definition

A commitment scheme is (t, ε)-*equivocable* if for any t-time adversary $\mathcal{A}$

$$\mathsf{Adv}^{\mathsf{eqv}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \enspace ,$$

where

$\mathcal{G}_0^{\mathcal{A}}$

$$\left[\begin{array}{l} \mathsf{pk} \leftarrow \mathsf{Gen} \\ \text{repeat} \\ \quad \left| \begin{array}{l} m_i \leftarrow \mathcal{A} \\ (c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m) \\ \text{Give } (c, d) \text{ to } \mathcal{A} \end{array} \right. \\ \text{until } m_i = \perp \\ \textbf{return } \mathcal{A} \end{array}\right.$$

$\mathcal{G}_1^{\mathcal{A}}$

$$\left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}^* \\ \text{repeat} \\ \quad \left| \begin{array}{l} (c, \sigma) \leftarrow \mathsf{Com}^*_{\mathsf{sk}}, m_i \leftarrow \mathcal{A} \\ d \leftarrow \mathsf{Equiv}_{\mathsf{sk}}(c, \sigma, m) \\ \text{Give } (c, d) \text{ to } \mathcal{A} \end{array} \right. \\ \text{until } m_i = \perp \\ \textbf{return } \mathcal{A} \end{array}\right.$$

A famous example

The Pedersen is perfectly equivocable commitment.

▷ **Setup.** Generate $x \leftarrow \mathbb{Z}_q^*$ and set $y \leftarrow g^x$.

▷ **Fake commitment.** Generate $s \leftarrow \mathbb{Z}_q$ and output $\hat{c} \leftarrow g^s$.

▷ **Equivocation.** To open $\hat{c}$, compute $r \leftarrow (s - m) \cdot x^{-1}$.

Proof

▷ Commitment value c has uniform distribution.

▷ For fixed c and m, there exists a unique value of r.

Equivocation leads to perfect simulation of (c, d) pairs.

Homomorphic commitments

A commitment scheme is $\otimes$-*homomorphic* if there exists an efficient coordinate-wise multiplication operation $\cdot$ defined over $\mathcal{C}$ and $\mathcal{D}$ such that

$$\mathsf{Com}_{\mathsf{pk}}(m_1) \cdot \mathsf{Com}_{\mathsf{pk}}(m_2) \equiv \mathsf{Com}_{\mathsf{pk}}(m_1 \otimes m_2) \enspace ,$$

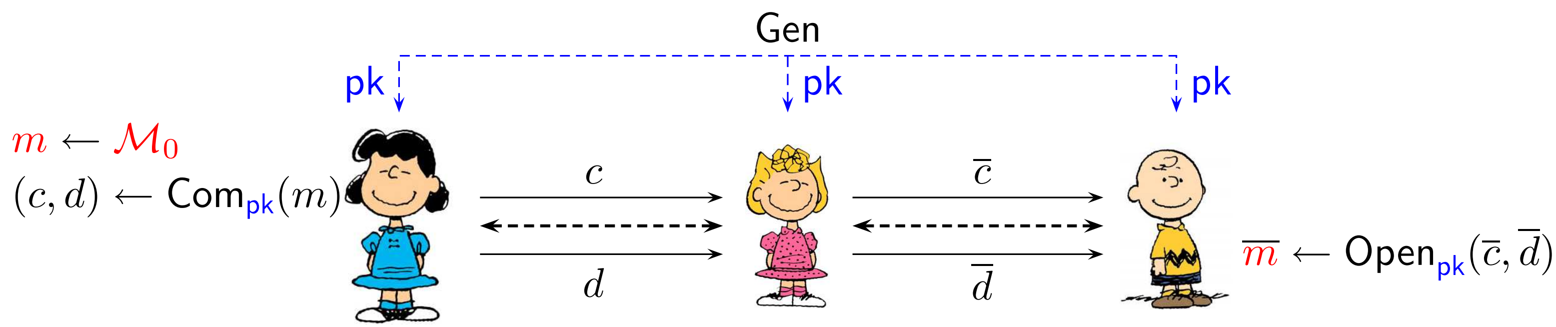
where the distributions coincide even if $\mathsf{Com}_{\mathsf{pk}}(m_1)$ is fixed.

Examples

- ElGamal commitment scheme
- Pedersen commitment scheme

Active Attacks

Non-malleability wrt opening



A commitment scheme is non-malleable wrt. opening if an adversary who knows the input distribution $\mathcal{M}_0$ cannot alter commitment and decommitment values c, d on the fly so that

- $\mathcal{A}$ cannot *efficiently* open the altered commitment value $\overline{c}$ to a message $\overline{m}$ that is related to original message m.

Commitment c does not help the adversary to create other commitments.

Formal definition

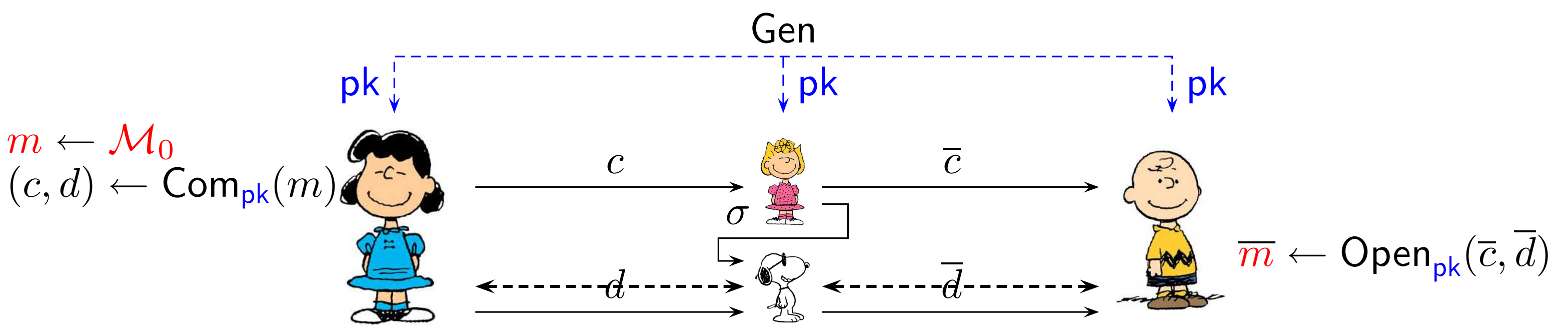
$\mathcal{G}_0^{\mathcal{A}}$

$$
\left[\begin{array}{l}
\mathsf{pk} \leftarrow \mathsf{Gen} \\
\mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\
m \leftarrow \mathcal{M}_0 \\
(c, d) \leftarrow \mathsf{Com}_{\mathsf{pk}}(m) \\
\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n \leftarrow \mathcal{A}(c) \\
\hat{d}_1, \ldots \hat{d}_n \leftarrow \mathcal{A}(d) \\
\text{if } c \in \{\hat{c}_1, \ldots, \hat{c}_n\} \text{ then } \textbf{return } 0 \\
\hat{m}_i \leftarrow \mathsf{Open}_{\mathsf{pk}}(\hat{c}_i, \hat{d}_i) \textbf{ for } i = 1, \ldots, n \\
\textbf{return } \pi(m, \hat{m}_1, \ldots, \hat{m}_n)
\end{array}\right.
$$

$\mathcal{G}_1^{\mathcal{A}}$

$$
\left[\begin{array}{l}
\mathsf{pk} \leftarrow \mathsf{Gen} \\
\mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\
m \leftarrow \mathcal{M}_0, \overline{m} \leftarrow \mathcal{M}_0 \\
(\overline{c}, \overline{d}) \leftarrow \mathsf{Com}_{\mathsf{pk}}(\overline{m}) \\
\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n \leftarrow \mathcal{A}(\overline{c}) \\
\hat{d}_1, \ldots \hat{d}_n \leftarrow \mathcal{A}(\overline{d}) \\
\text{if } c \in \{\hat{c}_1, \ldots, \hat{c}_n\} \text{ then } \textbf{return } 0 \\
\hat{m}_i \leftarrow \mathsf{Open}_{\mathsf{pk}}(\hat{c}_i, \hat{d}_i) \textbf{ for } i = 1, \ldots, n \\
\textbf{return } \pi(m, \hat{m}_1, \ldots, \hat{m}_n)
\end{array}\right.
$$

Non-malleability wrt commitment



A commitment scheme is non-malleable wrt commitment if an adversary $\mathcal{A}_1$ who knows the input distribution $\mathcal{M}_0$ cannot alter the commitment value c on the fly so that

- ▷ an unbounded adversary $\mathcal{A}_2$ cannot open the altered commitment value $\overline{c}$ to a message $\overline{m}$ that is related to original message m.

Commitment c does not help the adversary to create other commitments even if some secret values are leaked after the creation of c and $\overline{c}$.

Homological classification

$$
\begin{tikzcd}
\text{NM2-CPA} \arrow[d, Rightarrow] & \text{NM2-CCA1} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \text{NM2-CCA2} \arrow[l, Rightarrow] \arrow[d, Leftrightarrow, "?"'] \\
\text{NM1-CPA} \arrow[d, Rightarrow] & \text{NM1-CCA1} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \text{NM1-CCA2} \arrow[l, Rightarrow] \arrow[d, Leftrightarrow, "?"'] \\
\text{IND-CPA} & \text{IND-CCA1} \arrow[l, Rightarrow] & \text{IND-CCA2} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Can we define decommitment oracles such that the graph depicted above captures relations between various notions where

- ▷ NM1-XXX denotes non-malleability wrt opening,
- ▷ NM2-XXX denotes non-malleability wrt commitment.