MTAT.07.003 Cryptology II

Public Key Cryptosystems

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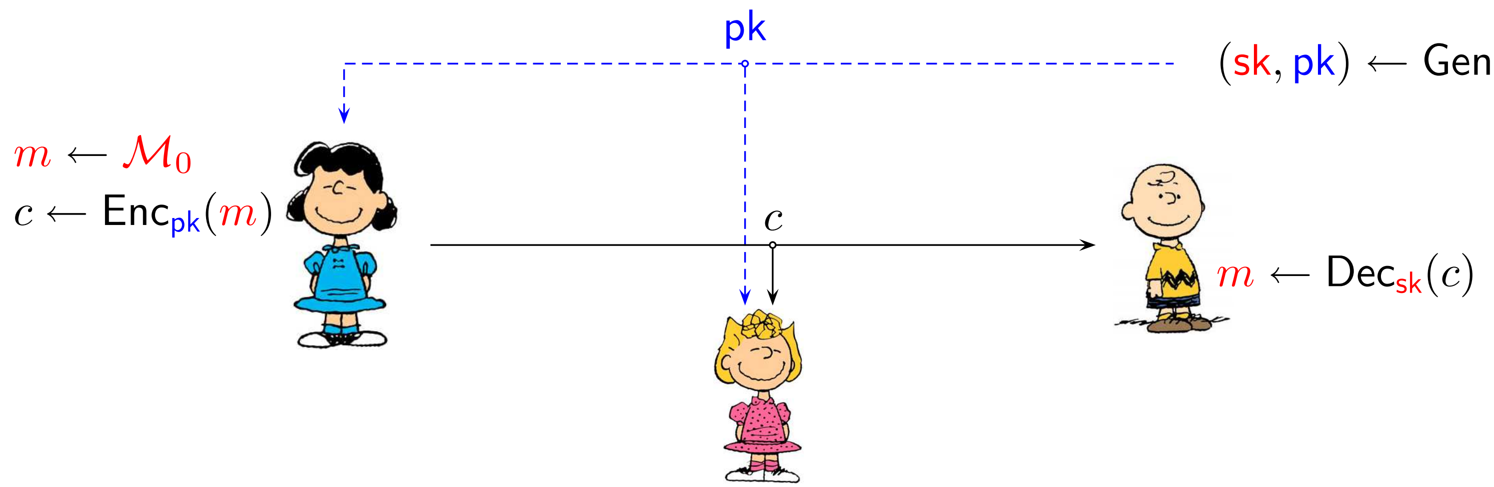
Formal Syntax

Public key cryptosystem



- A randomised *key generation algorithm* outputs a *secret key* sk and a *public key* pk. A public key gives ability to encrypt messages.
- A randomised *encryption algorithm* $\mathsf{Enc}_{\mathsf{pk}} : \mathcal{M} \to \mathcal{C}$ takes in a *plaintext* and outputs a corresponding *ciphertext*.
- A *decryption algorithm* $\mathsf{Dec}_{\mathsf{sk}} : \mathcal{C} \to \mathcal{M} \cup \{\perp\}$ recovers the plaintext or a special abort symbol $\perp$ to indicate invalid ciphertexts.

Example. RSA-1024 cryptosystem

Key generation Gen**:**

1. Choose uniformly 512-bit prime numbers p and q.
2. Compute $N = p \cdot q$ and $\phi(N) = (p-1)(q-1)$.
3. Choose uniformly $e \leftarrow \mathbb{Z}^*_{\phi(N)}$ and set $d = e^{-1} \mod \phi(N)$.
4. Output $\mathsf{sk} = (p, q, e, d)$ and $\mathsf{pk} = (N, e)$.

Encryption and decryption:

$$\mathcal{M} = \mathbb{Z}_N, \quad \mathcal{C} = \mathbb{Z}_N, \quad \mathcal{R} = \emptyset$$

$$\mathsf{Enc}_{\mathsf{pk}}(m) = m^e \mod N \qquad \mathsf{Dec}_{\mathsf{sk}}(c) = c^d \mod N \ .$$

Semantic Security

IND-CPA security

As a potential adversary $\mathcal{A}$ can influence which messages are encrypted, we must model the corresponding effects in our attack model. A cryptosystem $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is *(t, ε)-IND-CPA secure* if for all t-time adversaries $\mathcal{A}$:

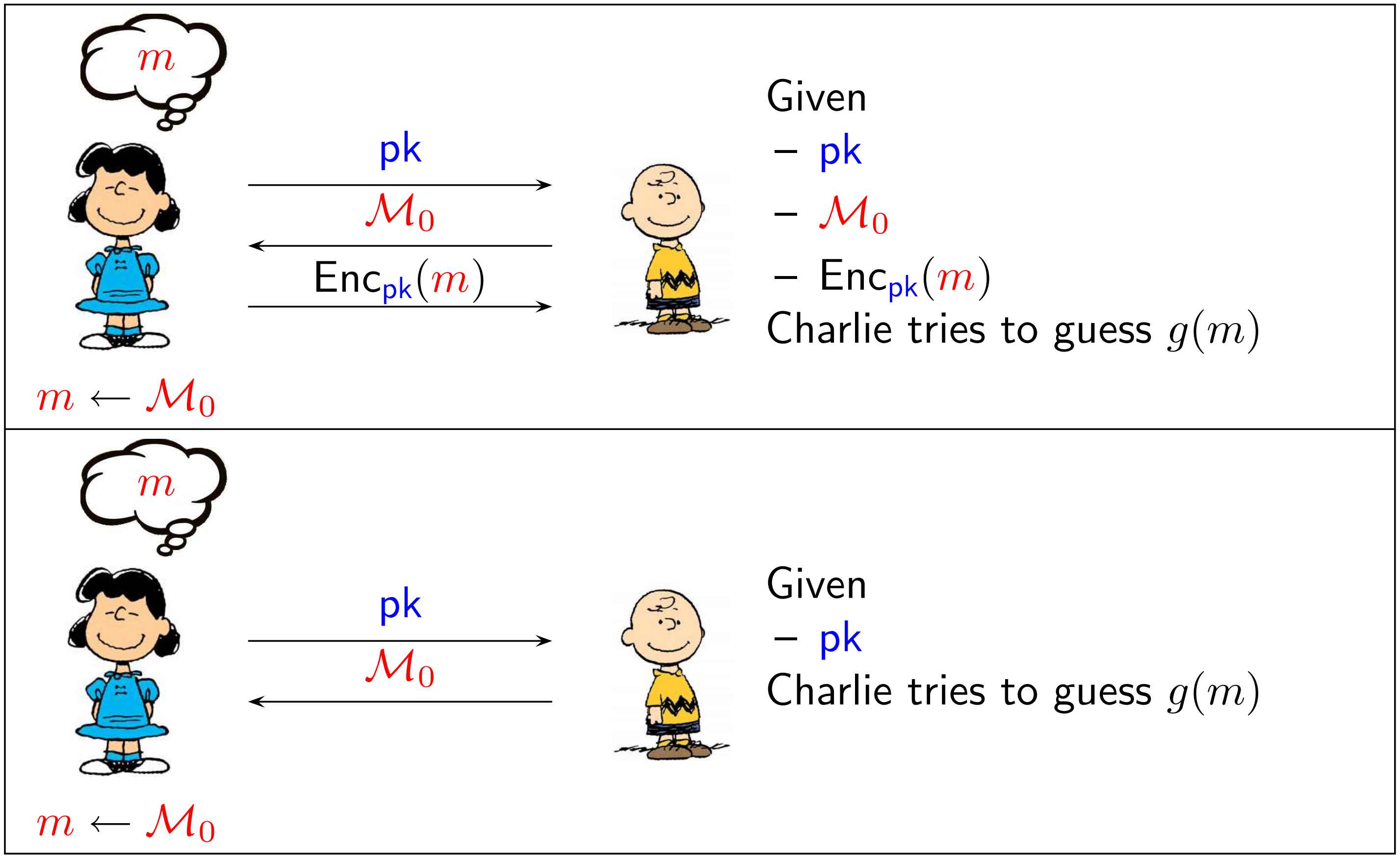
$$\mathsf{Adv}^{\text{ind-cpa}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \ ,$$

where the security games are defined as follows

$$\mathcal{G}_0^{\mathcal{A}} \qquad\qquad\qquad\qquad \mathcal{G}_1^{\mathcal{A}}$$

$$\left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}(\mathsf{Enc}_{\mathsf{pk}}(m_0)) \end{array}\right. \qquad \left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}(\mathsf{Enc}_{\mathsf{pk}}(m_1)) \end{array}\right.$$

Semantic security against adaptive influence



Formal definition

Consider following games:

$$\mathcal{G}_0^{\mathcal{A}}\left[\begin{array}{l}(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ \mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\ m \leftarrow \mathcal{M}_0 \\ c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m) \\ \textbf{return } [g(m) \stackrel{?}{=} \mathcal{A}(c)]\end{array}\right.$$

$$\mathcal{G}_1^{\mathcal{A}}\left[\begin{array}{l}(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ \mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\ m \leftarrow \mathcal{M}_0, \boxed{\overline{m} \leftarrow \mathcal{M}_0} \\ \boxed{\overline{c} \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\overline{m})} \\ \textbf{return } [g(m) \stackrel{?}{=} \boxed{\mathcal{A}(\overline{c})}]\end{array}\right.$$

The true guessing advantage is

$$\mathsf{Adv}_g^{\mathsf{sem}}(\mathcal{A}) = \Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right] \quad .$$

IND-CPA $\Rightarrow$ SEM-CPA

Theorem. Assume that g is a t_g-time function and it is always possible to obtain a sample from $\mathcal{M}_0$ in time t_m. Now if the cryptosystem is (t, ε)-IND-CPA secure, then for all $(t - t_g - 2t_m)$-time adversaries $\mathcal{A}$:

$$\mathsf{Adv}_g^{\mathsf{sem}}(\mathcal{A}) \leq \varepsilon \enspace .$$

Note that

▷ The function g might be randomised.

▷ The function g must be efficiently computable.

▷ The distribution $\mathcal{M}_0$ must be efficiently samplable.

An Example of IND-CPA Secure Cryptosystem

ElGamal cryptosystem

Combine the Diffie-Hellman key exchange protocol

Alice		**Bob**
$x \leftarrow \mathbb{Z}_{\lvert\mathbb{G}\rvert}$	$\xrightarrow{y=g^x}$	$k \leftarrow \mathbb{Z}_{\lvert\mathbb{G}\rvert}$
	$\xleftarrow{g^k}$	
$g^{xk} = (g^k)^x$		$g^{xk} = (g^x)^k$

with one-time pad by multiplication using in $\mathbb{G} = \langle g \rangle$ as encoding rule

$$\mathsf{Enc}_{\mathsf{pk}}(m) = (g^k, m \cdot g^{xk}) = (g^k, m \cdot y^k) \qquad \text{for all elements } m \in \mathbb{G}$$

with a public key $\mathsf{pk} = y = g^x$ and a secret key $\mathsf{sk} = x$.

Decisional Diffie-Hellman Assumption (DDH)

Definition. We say that a q-element multiplicative group $\mathbb{G}$ is (t, ε)-Decisional Diffie-Hellman group if for all t-time adversaries $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon$$

where the security games are defined as follows

$$\begin{array}{ll} \mathcal{G}_0^{\mathcal{A}} & \mathcal{G}_1^{\mathcal{A}} \\ \left[\begin{array}{l} x, k \leftarrow \mathbb{Z}_q \\ \textbf{return } \mathcal{A}(g, g^x, g^k, g^{xk}) \end{array}\right. & \left[\begin{array}{l} x, k, c \leftarrow \mathbb{Z}_q \\ \textbf{return } \mathcal{A}(g, g^x, g^k, g^c) \end{array}\right. \end{array}$$

The Diffie-Hellman key exchange protocol is secure under the DDH assumption, as an attacker cannot distinguish values g^{xk} and g^c.

DDH $\Rightarrow$ IND-CPA

Theorem. Let $\mathbb{G}$ be a (t, ε)-DDH group. Then the corresponding instantiation of the ElGamal cryptosystem is $(t, 2\varepsilon)$-IND-CPA secure.

Let $\mathcal{B}$ be good against IND-CPA games. Then we can consider the following algorithm $\mathcal{A}$:

1. Given (g, g^x, g^k, z), set pk $= g^x$ and $(m_0, m_1) \leftarrow \mathcal{B}(\mathsf{pk})$.
2. Toss a fair coin $b \leftarrow \{0, 1\}$ and set $c = (g^k, m_b z)$.
3. If $b \stackrel{?}{=} \mathcal{A}(c)$ return 1 else output 0.

We argue that this is a good strategy to win the DDH game:

- In the game $\mathcal{G}_0$, we simulate the bit guessing game.
- In the game $\mathcal{G}_1$, the guess guess is independent form b.

Hybrid encryption

Assume that $(\mathsf{Gen}, \mathsf{Enc}, \mathsf{Dec})$ is a IND-CPA secure public key cryptosystem and $(\mathsf{Gen}^\circ, \mathsf{Enc}^\circ, \mathsf{Dec}^\circ)$ is a IND-CPA secure symmetric key cryptosystem. Then we can construct a hybrid IND-CPA secure cryptosystem.

Key generation. Output the original secret and public key $(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen}$.

Encryption. For $m \in \mathcal{M}^\circ$ generate a session key $\mathsf{sk}^\circ \leftarrow \mathsf{Gen}^\circ$ and compute

$$\mathsf{Enc}^*_{\mathsf{pk}}(m) = (\mathsf{Enc}_{\mathsf{pk}}(\mathsf{sk}^\circ), \mathsf{Enc}^\circ_{\mathsf{sk}^\circ}(m))$$

Decryption. Given (c_1, c_2) compute $\mathsf{sk}^\circ \leftarrow \mathsf{Dec}_{\mathsf{sk}}(c_1)$ and output $\mathsf{Dec}^\circ_{\mathsf{sk}^\circ}(c_2)$.

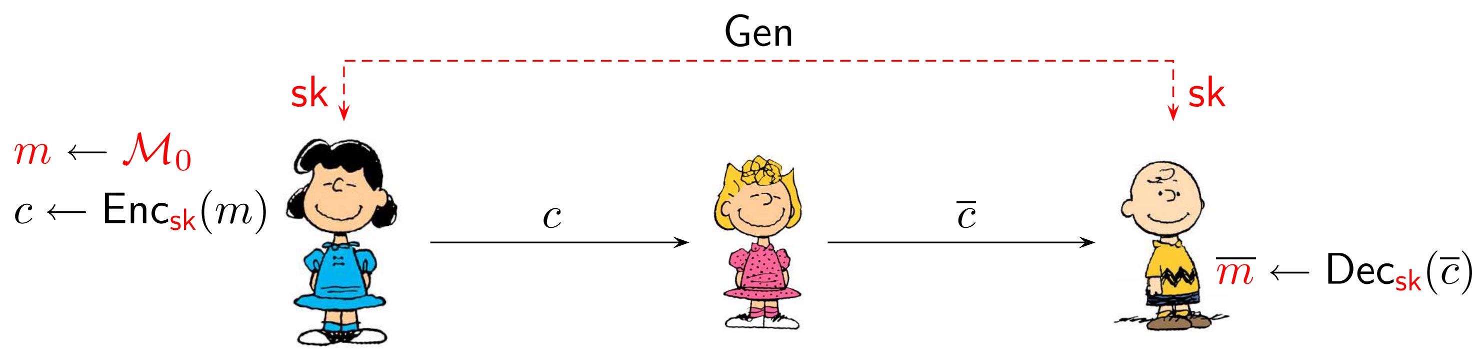
Theorem. The hybrid encryption is $(t, 2\varepsilon_1 + \varepsilon_2)$-IND-CPA secure if the public key cryptosystem is (t, ε_1)-IND-CPA secure and the symmetric key cryptosystem is (t, ε_2)-IND-CPA secure.

Corresponding proof

$$\mathcal{G}_0^{\mathcal{A}}\left[\begin{array}{l}
(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\
(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\
\mathsf{sk}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
\overline{\mathsf{sk}}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
\boxed{c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\mathsf{sk}^{\circ})} \\
c_2 \leftarrow \mathsf{Enc}^{\circ}_{\mathsf{sk}^{\circ}}(m_0) \\
\textbf{return } \mathcal{A}(c_1, c_2)
\end{array}\right.
\quad \xRightarrow[\varepsilon_1]{\text{IND-CPA}} \quad
\mathcal{G}_2^{\mathcal{A}}\left[\begin{array}{l}
(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\
(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\
\mathsf{sk}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
\overline{\mathsf{sk}}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
\boxed{c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\overline{\mathsf{sk}}^{\circ})} \\
\boxed{c_2 \leftarrow \mathsf{Enc}^{\circ}_{\mathsf{sk}^{\circ}}(m_0)} \\
\textbf{return } \mathcal{A}(c_1, c_2)
\end{array}\right.
\quad \xRightarrow[\varepsilon_2]{\text{IND-CPA}} \quad
\mathcal{G}_3^{\mathcal{A}}\left[\begin{array}{l}
(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\
(m_0, m_1) \leftarrow \mathcal{A}(\mathsf{pk}) \\
\mathsf{sk}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
\overline{\mathsf{sk}}^{\circ} \leftarrow \mathsf{Gen}^{\circ} \\
c_1 \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\overline{\mathsf{sk}}^{\circ}) \\
\boxed{c_2 \leftarrow \mathsf{Enc}^{\circ}_{\mathsf{sk}^{\circ}}(m_1)} \\
\textbf{return } \mathcal{A}(c_1, c_2)
\end{array}\right.$$

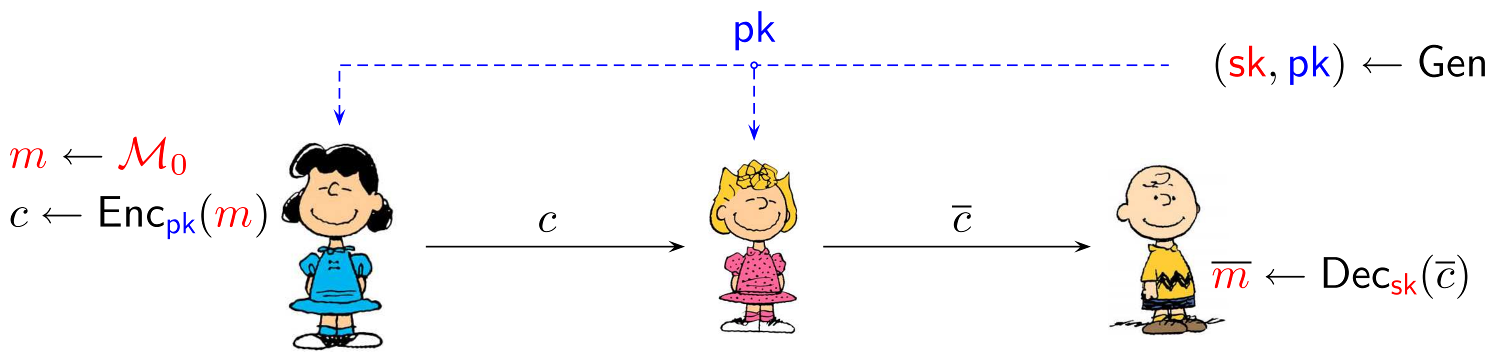
Ciphertext modification attacks

Symmetric key cryptosystem



- ▷ A malicious participant may control the communication network and alter the ciphertexts to bypass various security checks.
- ▷ A malicious participant may interact with a key holder and use him or her as an *encryption* or *decryption* oracle.
- ▷ A non-malleable encryption detects modifications in ciphertexts (*authenticated encryption*) or assures that m and $\overline{m}$ are unrelated.

Public key cryptosystem



- ▷ Active attacks are similar for public key cryptosystems. Except there is no need for encryption oracle, since the adversary knows the public key.
- ▷ Common cryptosystems detect tampered ciphertexts with high probability and thus the adversary cannot use the decryption oracle for useful tasks.

Homological classification

$$\begin{tikzcd}
\text{NM-CPA} \arrow[d, Rightarrow] & \text{NM-CCA1} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \text{NM-CCA2} \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] \\
\text{IND-CPA} & \text{IND-CCA1} \arrow[l, Rightarrow] & \text{IND-CCA2} \arrow[l, Rightarrow]
\end{tikzcd}$$

The figure above depicts the relations among various security properties of public key cryptosystems. In practise one normally needs:

▷ semantic security that follows IND-CPA security,

▷ safety against improper usage that follows form IND-CCA1 security,

▷ non-malleability of ciphertexts that follows form NM-CPA security.

Safety against improper usage

Cleverly crafted ciphertexts or ciphertext-like messages may provide relevant information about the secret key or even reveal the secret key.

Such attacks naturally occur in:

▷ smart card cracking (Satellite TV, TPM-modules, ID cards)

▷ authentication protocols (challenge-response protocols)

▷ side-channel attacks (timing information, encryption failures)

Minimal security level:

▷ Attacks reveal information only about currently known ciphertexts.

Affected cryptosystems:

– Rabin cryptosystem, some versions of NTRU cryptosystem, etc.

IND-CCA1 security

A cryptosystem is *(t, ε)-IND-CCA1 secure* if for all t-time adversaries $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{ind\text{-}cca1}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \ ,$$

where the security games are defined as follows

$$\begin{array}{ll} \mathcal{G}_0^{\mathcal{A}} & \mathcal{G}_1^{\mathcal{A}} \\ \left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}(\mathsf{Enc}_{\mathsf{pk}}(m_0)) \end{array}\right. & \left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}(\mathsf{Enc}_{\mathsf{pk}}(m_1)) \end{array}\right. \end{array}$$

and the oracle $\mathcal{O}_1$ serves decryption queries, i.e., $\mathcal{O}_1(c) = \mathsf{Dec}_{\mathsf{sk}}(c)$.

Rabin cryptosystem

Key generation Gen:

1. Choose uniformly 512-bit prime numbers p and q.
2. Compute $N = p \cdot q$ and $\phi(N) = (p-1)(q-1)$.
3. Output $\mathsf{sk} = (p, q)$ and $\mathsf{pk} = N$.

Encryption and decryption:

$$\mathcal{M} = \mathbb{Z}_N, \quad \mathcal{C} = \mathbb{Z}_N, \quad \mathcal{R} = \emptyset$$

$$\mathsf{Enc}_{\mathsf{pk}}(m) = m^2 \mod N \qquad \mathsf{Dec}_{\mathsf{sk}}(c) = \sqrt{c} \mod N \enspace .$$

Lunchtime attack

1. Choose $x \leftarrow \mathbb{Z}_N^*$ and set $c \leftarrow x^2 \mod N$.
2. Compute decryption $\overline{x} \leftarrow \mathcal{O}_1(c)$.
3. If $\overline{x} \neq \pm x$ then
   - Compute nontrivial square root $\xi = \overline{x} \cdot x^{-1} \mod N$
   - Compute a nontrivial factors $p \leftarrow \gcd(N, \xi + 1)$ and $q = N/p$.
   - Output a secret key sk $= (p, q)$.
4. Continue from Step 1.

Efficiency analysis

- Each iteration fails with probability $\frac{1}{2}$.
- With 80 decryption queries the failure probability is 2^{-80}.

IND-CCA2 security

A cryptosystem is *(t, ε)-IND-CCA2 secure* if for all t-time adversaries $\mathcal{A}$:

$$\mathsf{Adv}^{\mathsf{ind\text{-}cca1}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right| \leq \varepsilon \enspace ,$$

where the security games are defined as follows

$$\begin{array}{ll}
\mathcal{G}_0^{\mathcal{A}} & \mathcal{G}_1^{\mathcal{A}} \\
\left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}^{\mathcal{O}_2(\cdot)}(\mathsf{Enc}_{\mathsf{pk}}(m_0)) \end{array}\right. &
\left[\begin{array}{l} (\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}^{\mathcal{O}_2(\cdot)}(\mathsf{Enc}_{\mathsf{pk}}(m_1)) \end{array}\right.
\end{array}$$

and oracles $\mathcal{O}_1$ and $\mathcal{O}_2$ serve decryption queries, i.e., $\mathcal{O}_1(c) = \mathsf{Dec}_{\mathsf{sk}}(c)$ and $\mathcal{O}_2(c) = \mathsf{Dec}_{\mathsf{sk}}(c)$ for all non-challenge ciphertexts.

IND-CCA2 secure cryptosystems

All known IND-CCA2 secure cryptosystems include a non-interactive proof that the creator of the ciphertexts c knows the corresponding message m:

– the RSA-OAEP cryptosystem in the random oracle model,

– the Cramer-Shoup cryptosystem in standard model,

– the Kurosawa-Desmedt key encapsulation scheme.

Non-malleability

NM-CPA security

$m \leftarrow \mathcal{M}_0$

$\xrightarrow{\mathsf{pk}}$

$\xleftarrow{\mathcal{M}_0}$

$\xrightarrow{\mathsf{Enc}_{\mathsf{pk}}(m)}$

$\xleftarrow{\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n}$

Given
- pk
- $\mathcal{M}_0$
- $\mathsf{Enc}_{\mathsf{pk}}(m)$

Charlie tries to construct a predicate $\pi(\cdot)$ such that

$$\pi(m, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_n)) = 1$$

$m \leftarrow \mathcal{M}_0$

$\xrightarrow{\mathsf{pk}}$

$\xleftarrow{\mathcal{M}_0}$

$\xleftarrow{\pi(\cdot), \hat{c}_1, \ldots, \hat{c}_n}$

Given
- pk

Charlie tries to construct a predicate $\pi(\cdot)$ such that

$$\pi(m, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_n)) = 1$$

Formal definition

$\mathcal{G}_0^{\mathcal{A}}$

$$\left[\begin{array}{l}(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ \mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\ m \leftarrow \mathcal{M}_0 \\ c \leftarrow \mathsf{Enc}_{\mathsf{pk}}(m) \\ \pi(\cdot), \hat{c}_1, \ldots \hat{c}_n \leftarrow \mathcal{A}(c) \\ \text{if } c \in \{\hat{c}_1, \ldots \hat{c}_n\} \text{ then } \textbf{return } 0 \\ \textbf{return } \pi(m, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots)\end{array}\right.$$

$\mathcal{G}_1^{\mathcal{A}}$

$$\left[\begin{array}{l}(\mathsf{sk}, \mathsf{pk}) \leftarrow \mathsf{Gen} \\ \mathcal{M}_0 \leftarrow \mathcal{A}(\mathsf{pk}) \\ m \leftarrow \mathcal{M}_0, \boxed{\overline{m} \leftarrow \mathcal{M}_0} \\ \boxed{\overline{c} \leftarrow \mathsf{Enc}_{\mathsf{pk}}(\overline{m})} \\ \pi(\cdot), \hat{c}_1, \ldots \hat{c}_n \leftarrow \mathcal{A}(\overline{c}) \\ \text{if } \overline{c} \in \{\hat{c}_1, \ldots \hat{c}_n\} \text{ then } \textbf{return } 0 \\ \textbf{return } \pi(m, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots)\end{array}\right.$$

The true advantage is

$$\mathsf{Adv}^{\mathsf{nm\text{-}cpa}}(\mathcal{A}) = \left|\Pr\left[\mathcal{G}_0^{\mathcal{A}} = 1\right] - \Pr\left[\mathcal{G}_1^{\mathcal{A}} = 1\right]\right|$$

Homological classification

$$
\begin{tikzcd}
\text{NM-CPA} \arrow[d, Rightarrow] & \text{NM-CCA1} \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \text{NM-CCA2} \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] \\
\text{IND-CPA} & \text{IND-CCA1} \arrow[l, Rightarrow] & \text{IND-CCA2} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Horizontal implications are trivial.

- The adversary just gets more powerful in the row.

Downwards implications are trivial.

- A guess guess can be passed as a predicate $\pi(\cdot) \equiv 0$ and $\pi(\cdot) \equiv 1$.

IND-CCA2 $\Rightarrow$ NM-CC2

Theorem. Assume that $\pi(\cdot)$ is always a t_π-time predicate and it is always possible to obtain a sample from $\mathcal{M}_0$ in time t_m. Now if the cryptosystem is (t, ε)-IND-CCA2 secure, then for all $(t - t_g - 2t_m)$-time adversaries $\mathcal{A}$:

$$\mathsf{Adv}^{\text{nm-cca2}}(\mathcal{A}) \leq \varepsilon \enspace .$$

Note that

- ▷ The predicate $\pi(\cdot)$ might be randomised.
- ▷ The predicate $\pi(\cdot)$ might have variable number of arguments.
- ▷ The predicate $\pi(\cdot)$ must be a computationally efficient function.
- ▷ The distribution $\mathcal{M}_0$ must be efficiently samplable.

The corresponding proof

Let $\mathcal{B}$ be an adversary that is good in NM-CCA2 games. Then we can emulate NM-CCA2 game given access to the decryption oracle $\mathcal{O}_2$:

1. $\mathcal{A}$ forwards pk to $\mathcal{B}$ who sends back a description of $\mathcal{M}_0$.
2. $\mathcal{A}$ independently samples $m_0 \leftarrow \mathcal{M}_0$ and $m_1 \leftarrow \mathcal{M}_0$.
3. $\mathcal{A}$ forwards the challenge $\mathsf{Enc}_{\mathsf{pk}}(m_b)$ to $\mathcal{B}$.
4. $\mathcal{B}$ sends $\hat{c}_1, \ldots, \hat{c}_n$ and $\pi(\cdot)$ to $\mathcal{A}$ who
   - uses $\mathcal{O}_2$ to recover $\mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_n)$,
   - outputs $\pi(m_0, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_1), \ldots, \mathsf{Dec}_{\mathsf{sk}}(\hat{c}_n))$ as the final output.

Running time

The running time of $\mathcal{A}$ is $t_b + t_g + 2t_m$ where t_b is the running time of $\mathcal{B}$.

Further analysis by code rewriting

For clarity, let $\mathcal{Q}_0$ and $\mathcal{Q}_1$ denote the IND-CCA2 security games and $\mathcal{G}_0$ and $\mathcal{G}_1$ NM-CCA2 security games. Then note

$$\mathcal{Q}_0^{\mathcal{A}} \equiv \mathcal{G}_0^{\mathcal{B}} \qquad \text{and} \qquad \mathcal{Q}_1^{\mathcal{A}} \equiv \mathcal{G}_1^{\mathcal{B}}$$

where

$$\mathcal{Q}_0^{\mathcal{A}}$$

$$\left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}^{\mathcal{O}_2(\cdot)}(\mathsf{Enc}_{\mathsf{pk}}(m_0)) \end{array}\right.$$

$$\mathcal{Q}_1^{\mathcal{A}}$$

$$\left[\begin{array}{l} (\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen} \\ (m_0, m_1) \leftarrow \mathcal{A}^{\mathcal{O}_1(\cdot)}(\mathsf{pk}) \\ \textbf{return } \mathcal{A}^{\mathcal{O}_2(\cdot)}(\mathsf{Enc}_{\mathsf{pk}}(m_1)) \end{array}\right.$$